Option Greeks [cont'd].

For any portfolio consisting of

- risk-free asset,
- continuous-dividend paying stock,

and/or

- European options on that stock,

its value @ time-t can be written as

$$v(s, t, r, \delta, \sigma)$$

the value f'tion

stands for the time-t stock price

Def'n.

$$\Delta(\dots) := \frac{\partial}{\partial s} v(\dots)$$

$$\Gamma(\dots) := \frac{\partial^2}{\partial s^2} v(\dots)$$

$$\Theta(\dots) := \frac{\partial}{\partial t} v(\dots)$$

$$\rho(\dots) := \frac{\partial}{\partial r} v(\dots)$$

$$\psi(\dots) := \frac{\partial}{\partial \delta} v(\dots)$$

$$\text{vega}(\dots) := \frac{\partial}{\partial \sigma} v(\dots)$$

Example. Consider a zero-coupon bond w/ redemption amount of \$1 and maturity @ time T.

If this is all you have in your portfolio, then the time-t value of your portfolio:

$$v(s, \underset{=}{t}, \underset{=}{r}, \delta, \sigma) = e^{-r\overbrace{(T-t)}^{\text{time to maturity}}}$$

t: valuation date

$$\Rightarrow \Delta(\dots) = 0 \;;\; \Gamma(\dots) = 0$$

$$\Theta(\dots) = \frac{\partial}{\partial t}\left(e^{-r(T-t)}\right) = re^{-r(T-t)} > 0$$

$$\rho(\dots) = \frac{\partial}{\partial r}\left(e^{-r(T-t)}\right) = -(T-t)e^{-r(T-t)} < 0$$

Example. Outright purchase of a non-dividend-paying stock.

$$\Rightarrow v(s, t, r, \delta, \sigma) = s$$

s: stands for the time-t stock price

$$\Rightarrow \Delta(\dots) = 1 \quad \Rightarrow \Gamma(\dots) = 0$$

other greeks $= 0$ w/ skipping ψ.

Example. Consider a prepaid forward contract on a continuous-dividend-paying stock.

$$v(s, t, r, \delta, \sigma) = s e^{-\delta \underbrace{(T-t)}_{\text{time to delivery}}}$$

$$\Rightarrow \quad \Delta(\dots) = e^{-\delta(T-t)} > 0 \ ; \quad \Gamma(\dots) = 0$$

$$\Theta(\dots) = \delta s e^{-\delta(T-t)} > 0$$

Example. A European call w/ strike K and exercise date T.

The Black-Scholes price is:

$$v_c(s, t, r, \delta, \sigma) =$$

$$= s e^{-\delta(T-t)} \cdot N(d_1(\dots)) - K e^{-r(T-t)} \cdot N(d_2(\dots))$$

w/ $$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left[\ln\left(\frac{s}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t) \right]$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

By def'n:

$$\Delta_c(\dots) = \frac{\partial}{\partial s} v_c(\dots)$$

After the chain rule & the product rule:

$$\Delta_c(s,t,r,\delta,\sigma) = e^{-\delta(T-t)} \cdot N(d_1(\dots)) > 0$$

The positivity makes sense since the call is long w.r.t. the underlying.

Example. A European K-strike, T-time put.

Put-Call Parity:

$$v_c(\dots) - v_p(\dots) = se^{-\delta(T-t)} - Ke^{-r(T-t)}$$

$\frac{\partial}{\partial s}$

$$\Delta_c(\dots) - \Delta_p(\dots) = e^{-\delta(T-t)}$$

$$\Rightarrow \Delta_p(\dots) = \Delta_c(\dots) - e^{-\delta(T-t)}$$

$$= e^{-\delta(T-t)} \cdot N(d_1(\dots)) - e^{-\delta(T-t)}$$

$$= e^{-\delta(T-t)}\left(N(d_1(\dots)) - 1\right)$$

$$= -e^{-\delta(T-t)} \cdot N(-d_1(\dots)) < 0$$

Recall: Puts are <u>short</u> w.r.t. the underlying.

FORMULAS FOR OPTION GREEKS:

Delta (Δ)

Call: $e^{-\delta(T-t)}N(d_1)$,

Put: $-e^{-\delta(T-t)}N(-d_1)$

Gamma (Γ)

Call and Put: $\dfrac{e^{-\delta(T-t)}N'(d_1)}{S\sigma\sqrt{T-t}}$

Theta (θ)

Call: $\delta Se^{-\delta(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2) - \dfrac{Ke^{-r(T-t)}N'(d_2)\sigma}{2\sqrt{T-t}}$,

Put: Call Theta + $rKe^{-r(T-t)} - \delta Se^{-\delta(T-t)}$

Vega

Call and Put: $Se^{-\delta(T-t)}N'(d_1)\sqrt{T-t}$

Rho (ρ)

Call: $(T-t)Ke^{-r(T-t)}N(d_2)$,

Put: $-(T-t)Ke^{-r(T-t)}N(-d_2)$

Psi (ψ)

Call: $-(T-t)Se^{-\delta(T-t)}N(d_1)$,

Put: $(T-t)Se^{-\delta(T-t)}N(-d_1)$

(A) 7.32 million

(B) 7.42 million

(C) 7.52 million

(D) 7.62 million

(E) 7.72 million

At home: Calculate the price!

$T = 1/4 \quad K = 41.5$

8. You are considering the purchase of a 3-month 41.5-strike American call option on a nondividend-paying stock. $\Rightarrow$ NEVER OPTIMALLY EXERCISED EARLY. $\Rightarrow$ SAME AS EUROPEAN

You are given:

(i) The Black-Scholes framework holds.

(ii) The stock is currently selling for 40. $S(0) = 40$

(iii) The stock's volatility is 30%. $\sigma = 0.3$

(iv) The current call option delta is 0.5.

$$\underbrace{\Delta_C(S(0), 0)}_{\substack{\parallel\ \delta = 0 \\ N(d_1(S(0),0))}} = 0.5$$

Determine the current price of the option.

(A) $20 - 20.453\int_{-\infty}^{0.15} e^{-x^2/2}dx$

(B) $20 - 16.138\int_{-\infty}^{0.15} e^{-x^2/2}dx$

(C) $20 - 40.453\int_{-\infty}^{0.15} e^{-x^2/2}dx$

(D) $16.138\int_{-\infty}^{0.15} e^{-x^2/2}dx - 20.453$

(E) $40.453\int_{-\infty}^{0.15} e^{-x^2/2}dx - 20.453$

The current option price:

$$v_C(S(0), 0) = \underset{\checkmark}{S(0)} \cdot \underset{\checkmark}{N(d_1(S(0),0))} - \underset{\checkmark}{K} e^{-r \cdot T} \cdot N(d_2(S(0),0))$$

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From (iv), we have

$$N(d_1(S(0),0)) = 0.5$$

$$\Rightarrow d_1(S(0),0) = 0$$

$$\Rightarrow \frac{1}{\sigma\sqrt{T}}\Big[\underbrace{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T}_{=0}\Big] = 0$$

$$\Rightarrow \left(r + \frac{0.09}{2}\right)\cdot\frac{1}{4} = -\ln\left(\frac{S(0)}{K}\right) = \ln\left(\frac{41.5}{40}\right)$$

$$\Rightarrow r = 4\cdot\ln\left(\frac{41.5}{40}\right) - 0.045 = 0.1023$$

$$d_2(S(0),0) = 0 - 0.3\sqrt{\tfrac{1}{4}} = -0.15$$

$$\Rightarrow v_c(S(0),0) = 40\cdot(0.5) - 41.5e^{-0.1023(1/4)}\cdot N(-0.15)$$

Recall:

$$N(x) = \int_{-\infty}^{x}\varphi(z)\,dz = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$\Rightarrow v_c(S(0),0) = 20 - 40.453\,(1 - N(0.15))$$

$$= 40.453\,N(0.15) - 20.453$$

$$= \frac{40.453}{\sqrt{2\pi}}\int_{-\infty}^{0.15}e^{-\frac{z^2}{2}}dz - 20.453 \Rightarrow (D)$$

$$16.138 = \frac{40.453}{\sqrt{2\pi}}$$

7.