The Ornstein-Uhlenbeck process

*Sample MFE Problem #24.*

\[ \frac{dX(t)}{dt} = \lambda (\alpha - X(t)) dt + \sigma dZ(t) \]

\(\text{w/ } Z \text{ a std BM}\)

\[ X(0) \text{... initial conditions} \]

\[ \frac{dX(t)}{dt} = \lambda \alpha \ dt - \lambda X(t) \ dt + \sigma dZ(t) \]

\[ \left( \frac{dX(t)}{dt} + \lambda X(t) dt = \lambda \alpha dt + \sigma dZ(t) \right) \cdot e^{\lambda t} \]

\[ e^{\lambda t} \frac{dX(t)}{dt} + X(t) \lambda e^{\lambda t} dt = \lambda \alpha e^{\lambda t} dt + \sigma e^{\lambda t} dZ(t) \]

\[ d\left( X(t) e^{\lambda t} \right) = \alpha \lambda e^{\lambda t} dt + \sigma e^{\lambda t} dZ(t) \]

\[ X(t) e^{\lambda t} = \alpha \lambda \int_0^t e^{\lambda s} ds + \sigma \int_0^t e^{\lambda s} dZ(s) + X(0) \]

\[ \frac{1}{\lambda} \left. e^{\lambda s} \right|_{s=0}^t \]

\[ \frac{1}{\lambda} (e^{\lambda t} - 1) \]
\[ X(t) e^{\lambda t} = \alpha X(t) + \frac{1}{\lambda} (e^{\lambda t} - 1) + \sigma \int_0^t e^{\lambda s} dZ(s) + X(0) \]

\[ X(t) = e^{-\lambda t} X(0) + \alpha (1 - e^{-\lambda t}) + \sigma \int_0^t e^{\lambda (t-s)} dZ(s) \]

\[ E[X(t)] = e^{-\lambda t} X(0) + \alpha (1 - e^{-\lambda t}) \]

d\( X(t) = 2(4 - X(t))dt + 8dZ(t) \)

w/ \( Z \) a std BM

Set \( Y(t) = \frac{1}{X(t)} \).

\[
dY(t) = \alpha(Y(t))dt + \beta(Y(t))dZ(t)
\]

deterministic functions

Find \( \alpha(\frac{1}{2}) \).

Sol'n: Ito's Lemma: \( F(t,x) = \frac{1}{x} = x^{-1} \)

\[
\Rightarrow F_t = 0, F_x(t,x) = -x^{-2}, F_{xx}(t,x) = +2x^{-3}
\]

\[
\Rightarrow dY(t) = -[X(t)]^{-2}dX(t) + \frac{1}{2} \cdot 2 \cdot [X(t)]^{-3}dX(t)^2
\]

\[
\Rightarrow dY(t) = -[Y(t)]^2 \left( 2(4 - (Y(t))^{-1})dt + 8dZ(t) \right) + [Y(t)]^3 \frac{64}{3} dt
\]

\[
\Rightarrow \alpha(Y(t)) = - [Y(t)]^2 \cdot 2 \left( 4 - \frac{1}{Y(t)} \right) + 64 [Y(t)]^3
\]

\[
\Rightarrow \alpha(y) = -2y \left( 4y - 1 \right) + 64y^3
\]
\[ a(\frac{3}{2}) = -2 \cdot \frac{1}{2} \cdot (4 \cdot \frac{1}{2} - 1) + 64 \cdot \frac{1}{8} \]
\[ = -1 + 8 = 7 \Rightarrow (2). \]
The stochastic process \( \{R(t)\} \) is given by

\[
R(t) = R(0)e^{-t} + 0.05(1-e^{-t}) + 0.1\int_0^t e^{-s-t} \sqrt{R(s)}dZ(s),
\]

where \( \{Z(t)\} \) is a standard Brownian motion.

Define \( X(t) = [R(t)]^2 \).

Find \( dX(t) \).

(A) \[
0.1\sqrt{X(t)} - 2X(t) dt + 0.2[X(t)]^{3/2} dZ(t)
\]

(B) \[
0.11\sqrt{X(t)} - 2X(t) dt + 0.2[X(t)]^{3/2} dZ(t)
\]

(C) \[
0.12\sqrt{X(t)} - 2X(t) dt + 0.2[X(t)]^{3/2} dX(t)
\]

(D) \[
0.01 + [0.1 - 2R(0)]e^{-t} \sqrt{X(t)}dt + 0.2[X(t)]^{3/2} dZ(t)
\]

(E) \[
0.1 - 2R(0) e^{-t} \sqrt{X(t)}dt + 0.2[X(t)]^{3/2} dZ(t)
\]
\[ dX(t) = ? \]
\[
\downarrow
\]
Need Itô's Lemma for Itô processes.
\[
\downarrow
\]
Need \( dR(t) \).

Rewrite \( \star \) as:
\[
e^{t}R(t) = R(0) + 0.05(e^{t} - 1) + 0.1 \int_{0}^{t} e^{s\sqrt{R(s)}} dZ(s)
\]
\[
d(e^{t}R(t)) = 0.05e^{t}dt + 0.1 e^{t}\sqrt{R(t)} dZ(t)
\]
\[
R(t)e^{t}dt + e^{t}dR(t) = 0.05e^{t}dt + 0.1 e^{t}\sqrt{R(t)} dZ(t)
\]
\[
dR(t) = (0.05 - R(t))dt + 0.1 \sqrt{R(t)} dZ(t)
\]

\[
F(t, x) = x^2 \Rightarrow F_t = 0, \ F_x(t, x) = 2x, \ F_{xx}(t, x) = 2
\]

Itô's Lemma:
\[
dx(t) = dF(t, R(t)) =
\]
\[
= 2 \cdot R(t) dR(t) + \frac{1}{2} \cdot 2(dR(t))^2
\]
\[
dx(t) = 2 R(t) (0.05 - R(t))dt + 0.1 \sqrt{R(t)} dZ(t)
\]
\[
+ 0.01 \cdot R(t) dt
\]
\[ \frac{dx(t)}{dt} = (2R(t)(0.05 - R(t)) + 0.01 \cdot R(t)) \, dt \\
+ 0.2 \, R(t) \sqrt{R(t)} \, dZ(t) \]

\[ R(t) = \sqrt{x(t)} \]

\[ \text{dt-term:} \quad 0.1R(t) - 2(R(t))^2 + 0.01 \cdot R(t) \]

\[ = 0.11 \sqrt{x(t)} - 2x(t) \Rightarrow \square \]
Equilibrium Short-Rate Bond Price Models

- The Rendleman-Bartter model
- The Vasicek model
- The Cox-Ingersoll-Ross model
Equilibrium Short-Rate Bond Price Models

- We discuss three bond pricing models based on the equilibrium equation, in which all bond prices are driven by the short-term interest rate $r$:
  - The Rendleman-Bartter model
  - The Vasicek model
  - The Cox-Ingersoll-Ross model
- They differ in their specification of the coefficients of the SDE that the short-term interest rate is required to satisfy
An Overly Simple Model

- The simplest models of the short-term interest rate are those in which the interest rate is modeled as an arithmetic or a geometric Brownian motion.
- For example:
  \[ dr = a \, dt + \sigma dZ \]  
  \text{(ABM)}
- In this specification, the short-rate is normally distributed with mean \( r_0 + at \) and variance \( \sigma^2 t \).
- There are several shortcomings of this model:
  - The \( r \) can become negative.
  - The drift in the SDE for \( r \) is constant - meaning that for \( a > 0 \), \( r \) is expected to drift to \( \infty \) over a long time.
  - The volatility of the short-rate is the same whether the rate is high or low - this clashes with the practical observation that higher rates tend to be more volatile.
The Rendaleman-Bartter model

- The Rendaleman-Bartter model assumes that the short-rate follows a geometric Brownian motion, i.e., it satisfies the following SDE:

  \[ dr = ar \, dt + \sigma r \, dZ \]

- An objection to this model is that interest rates can be arbitrarily high. In practice, we would expect rates to exhibit mean reversion