**NAME:**

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
In-Term Exam I
Instructor: Milica Ćudina

**Notes:** This is a closed book and closed notes exam. The maximum number of points on this exam is 80.

**Time:** 50 minutes

**TRUE/FALSE**

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**MULTIPLE CHOICE**

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**FOR GRADER’S USE ONLY:**

| DEF’N | T/F | 1.9 | 1.10 | M.C. | Σ |
1.1. **TRUE/FALSE QUESTIONS.** Please note your answers on the front page.

**Problem 1.1.** (2 points) Under the risk-neutral probability measure, every option on a particular stock has the continuously compounded, risk-free interest rate as its mean rate of return. *True or false?*

**Solution:** TRUE

**Problem 1.2.** The Black-Scholes option pricing formula can *always* be used for pricing American-type call options on non-dividend-paying assets. *True or false?*

**Solution:** TRUE

**Problem 1.3.** The Black-Scholes option pricing formula can as a rule only be used for pricing European-type options. *True or false?*

**Solution:** TRUE

**Problem 1.4.** In developing the log-normal stock price model, we assumed normally distributed realized returns. *True or false?*

**Solution:** TRUE

**Problem 1.5.** The probability density function of standard normal distribution is even, i.e., its graph is symmetric around the vertical axis. *True or false?*

**Solution:** TRUE

**Problem 1.6.** Let $S(T)$ be the lognormal stock price. Then, the expected payoff of a time-$T$, strike-$K$ European put option is at least $(K - \mathbb{E}[S(T)])_+$. *True or false?*

**Solution:** TRUE

This is a straightforward application of Jensen’s inequality.

**Problem 1.7.** Assume the Black-Scholes framework for modeling the price of a continuous-dividend-paying stock. The mean stock price is a strictly increasing function of time regardless of the parameter values. *True or false?*

**Solution:** FALSE

**Problem 1.8.** (2 pts) In the setting of the Black-Scholes stock-price model, let $S = \{S(t), t \geq 0\}$ denote the stock price whose drift is $\alpha$ and volatility is $\sigma$.

Then,

$$
\mathbb{E}[S(t + h) \mid S(t)] = S(t), \text{ for every } t \geq 0 \text{ and } h > 0.
$$

*True or false?*

**Solution:** FALSE
1.2. FREE-RESPONSE PROBLEMS. Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.9. (20 points) Let $S(0) = $120, $K = $100, $\sigma = 0.3$, $r = 0$ and $\delta = 0.08$.

a. (10 pts) Let $V_C(0, T)$ denote the Black-Scholes European call price for the maturity $T$. Does the limit of $V_C(0, T)$ as $T \to \infty$ exist? If it does, what is it?

b. (8 pts) Now, set $r = 0.001$ and let $V_C(0, T, r)$ denote the Black-Scholes European call price for the maturity $T$. Again, how does $V_C(0, T, r)$ behave as $T \to \infty$?

c. (2 pts) Interpret in a sentence or two the differences, if any, between your answers to questions in a. and b.

Solution:

a. By the Black-Scholes pricing formula, the function $V_C(0, T)$ has the form

$$V_C(0, T) = S(0)e^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) = S(0)e^{-\delta T} N(d_1) - KN(d_2),$$

where $N$ denotes the distribution function of the unit normal distribution and

$$d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + \left( -\delta + \frac{1}{2} \sigma^2 \right) T \right],
\quad d_2 = d_1 - \sigma \sqrt{T}.$$

As $T \to \infty$, we have that

$$e^{-\delta T} N(d_1) \leq e^{-\delta T} \to 0,
\quad d_2 \to -\infty \Rightarrow N(d_2) \to 0.$$

Hence,

$$V_C(0, T) \to 0, \text{ as } T \to \infty.$$

b. In this case, the price of the call option reads as

$$V_C(0, T, r) = S e^{-\delta T} N(d_1) - K e^{-rT} N(d_2),$$

with

$$d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + \left( r - \delta + \frac{1}{2} \sigma^2 \right) T \right],
\quad d_2 = d_1 - \sigma \sqrt{T}.$$

As $T \to \infty$, we have that

$$e^{-\delta T} N(d_1) \leq e^{-\delta T} \to 0,
\quad d_2 \to -\infty \Rightarrow N(d_2) \to 0.$$

Since the function $N$ is bounded between 0 and 1, we see that as $T \to \infty$, $V_C(0, T, r) \to 0$. 
c. Until the call option is exercised, the owner of the option can earn interest on the strike price which he/she can invest at the risk-free rate. However, in forfeiting the physical ownership of the asset, he/she also forfeits the possible dividend payments. It would be interesting to see what happens for $\delta = 0$. 
Problem 1.10. (10 points) Source: CAS, Exam 8, Spring 2000, Problem #31.
You are considering the purchase of a three-month European put option on a stock with an announced dividend payment of $1.50 in two months. You are given the following information:

- The strike price is $50.
- The continuously compounded risk-free interest rate is 10%.
- The annual volatility of a three-month prepaid forward on the stock is 0.30.
- The stock follows the Black-Scholes framework.
- In our usual notation, $d_2 = -0.1086$.

Determine the current stock price.

Solution: Using the given value of $d_2$, we can solve for the prepaid forward price of the underlying stock.

$$-0.1086 = d_2 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{F_{0,T}^P(S)}{K} \right) + \left( r - \frac{\sigma^2}{2} \right) T \right]$$

$$= \frac{1}{0.3 \sqrt{\frac{1}{4}}} \left[ \ln \left( \frac{F_{0,T}^P(S)}{50} \right) + \left( 0.1 - \frac{0.3^2}{2} \right) \frac{1}{4} \right].$$

So,

$$-0.1086(0.15) = -0.01629 = \ln \left( \frac{F_{0,T}^P(S)}{50} \right) + (0.055) \frac{1}{4} = \ln \left( \frac{F_{0,T}^P(S)}{50} \right) + 0.01375.$$

Hence,

$$\frac{F_{0,T}^P(S)}{50} = e^{-0.03004} \Rightarrow F_{0,T}^P(S) = 50e^{-0.03004} = 48.52.$$

Finally,

$$S(0) = F_{0,T}^P(S) + 1.5e^{-0.10(\frac{1}{2})} = 48.52 + 1.5e^{-0.10(\frac{1}{2})} = 49.9955 \approx 50.$$
1.3. **MULTIPLE CHOICE QUESTIONS.** Please note your answers on the front page.

**Problem 1.11.** (5 points) Let $S = \{S(t), t \geq 0\}$ denote the stock-price process following a lognormal distribution. The mean stock price at time $-1$ equals 120 and the median stock price 115. What is the probability that the time $-1$ stock price exceeds 100?

(a) About 0.6844  
(b) About 0.7823  
(c) About 0.8529  
(d) About 0.8932  
(e) None of the above.

**Solution:** (a)

Since $S = \{S(t), t \geq 0\}$ is the stock-price process modeled by a geometric Brownian motion, the stock price at time $-1$ is lognormally distributed. In fact, using our usual parameters, we can rewrite it as

$$S(1) = S(0)e^{(\alpha - \delta - \frac{1}{2}\sigma^2) + \sigma Z(1)}.$$ 

Recall that the median of $S(1)$ equals $S(0)e^{(\alpha - \delta - \frac{1}{2}\sigma^2)}$. So, the required probability can be expressed as

$$\mathbb{P}[S(1) > 100] = \mathbb{P}[115e^{\sigma Z(1)} > 100] = \mathbb{P}\left[Z(1) > \frac{1}{\sigma}\ln\left(\frac{100}{115}\right)\right] = \mathbb{P}\left[Z(1) < \frac{1}{\sigma}\ln\left(\frac{115}{100}\right)\right] = N\left(\frac{1}{\sigma}\ln\left(\frac{115}{100}\right)\right).$$ 

Since the mean of $S(1)$ equals $S(0)e^{(\alpha - \delta)}$, we have

$$e^{\frac{1}{2}\sigma^2} = \frac{120}{115} \Rightarrow \sigma = \sqrt{2\ln(1.04348)} = 0.2918.$$ 

So, our final answer is

$$\mathbb{P}[S(1) > 100] = N(0.48) = 0.6844.$$ 

**Problem 1.12.** (5 pts) Assume the Black-Scholes framework for modeling the futures prices on oil. Let the futures price of oil for delivery in one year equal $90.00. The volatility of this price is given to be 0.26.

Assume that the continuously compounded risk-free interest rate equals 0.05.

Consider an at-the-money European put option on the above futures contract with the exercise date in one year.

What is the Black-Scholes price $V_P(0)$ of this put option?

(a) $6.62  
(b) $7.73  
(c) $8.85  
(d) $9.31  
(e) None of the above.
Solution: (c)
We are given the one-year futures price on oil to be $F_{0.1} = 90$.

Since the option in question is given to be at-the-money, it is immediate that in our usual notation,

\[ d_1 = \frac{1}{2} \sigma \sqrt{T} = \frac{0.25}{2} = 0.13, \quad d_2 = -d_1 = -0.13. \]

The price of the put option is

\[ V_P(0) = F_{0.1} e^{-rT} (2N(-d_1) - 1) = 90 e^{-0.05}(2 \times 0.5517 - 1) = 8.85. \]
Problem 1.13. IFM Sample (Introductory) Problem #6.
The following relates to one share of XYZ stock:
- The current price is 100.
- The forward price for delivery in one year is 105.
- An investor who decides to long the forward contract denotes by \( P \) the expected stock price in one year.

Determine which of the following statements about \( P \) is TRUE.

(A) \( P < 100 \)
(B) \( P = 100 \)
(C) \( 100 < P < 105 \)
(D) \( P = 105 \)
(E) \( P > 105 \)

Solution: \( \text{(e)} \)
Since the investor decided to long the forward contract, the payoff/profit will be

\[ S(T) - 105 \]

where \( S(T) \) denotes the stock price on the delivery date \( T \). The reason the investor chose to long the forward was the belief that the expected profit would be positive, i.e.,

\[ \mathbb{E}[S(T)] = P > 105. \]

Problem 1.14. (5 pts) Which of the following gives the correct values for the delta and gamma of a single share of non-dividend-paying stock?

(a) \( \Delta = 1, \Gamma = 1 \)
(b) \( \Delta = 1, \Gamma = 0 \)
(c) \( \Delta = 0, \Gamma = 1 \)
(d) \( \Delta = 0, \Gamma = 0 \)
(e) None of the above.

Solution: \( \text{(b)} \)
Problem 1.15. (5 points) Let the current price of a continuous-dividend-paying stock be denoted by $S(0)$. We model the time-$T$ stock price as lognormal. The mean rate of return on the stock is 0.10, its dividend yield is 0.01, and its volatility is 0.20. The continuously-compounded, risk-free interest rate is 0.03. You invest in one share of stock at time-$0$. and simultaneously deposit an amount $\varphi S(0)$ in a savings account. Assume continuous and immediate reinvestment of all dividends in the same stock. What should the proportion $\varphi$ be so that the VaR at the level 0.05 of your total wealth at time-$1$ equals today’s stock price $S(0)$?

(a) $\varphi = 0.0573$
(b) $\varphi = 0.1966$
(c) $\varphi = 0.2139$
(d) $\varphi = 0.5$
(e) None of the above.

Solution: (c)
The total wealth at time-$1$ is equal to $e^\delta S(1) + \varphi S(0)e^r$. So, our condition on the VaR is

$$P[e^\delta S(1) + \varphi S(0)e^r < S(0)] = 0.05.$$

The above is equivalent to

$$P[S(0)e^{\alpha - \frac{\sigma^2}{2} + \sigma Z} + \varphi S(0)e^r < S(0)] = 0.05$$

with $Z \sim N(0, 1)$. We have that the above is, in turn, equivalent to

$$P[e^{\alpha - \frac{\sigma^2}{2} + \sigma Z} + \varphi e^r < 1] = 0.05.$$

The constant $z^*$ such that $P[Z < z^*] = 0.05$ equals $-1.645$. Therefore,

$$e^{\alpha - \frac{\sigma^2}{2} + \sigma(-1.645)} + \varphi e^r = 1.$$

So,

$$\varphi = e^{-r} \left(1 - e^{\alpha - \frac{\sigma^2}{2} + \sigma(-1.645)}\right) = e^{-0.03} \left(1 - e^{0.10 - \frac{(0.2)^2}{2} + 0.2(-1.645)}\right) = 0.2139.$$

Problem 1.16. (5 points) The current exchange rate of the Bolivian Boliviano to the USD is $0.14$ per Bolivian Boliviano. The volatility of the exchange rate is given to be 0.10. The USD continuously compounded, risk-free interest rate is $r_S = 0.024$ while the Bolivian Boliviano continuously compounded, risk-free interest rate equals $r_B = 0.05$. Assume the Garman-Kohlhagen model is used for the exchange rate. Find the price of a one-year gap call option on the Bolivian Boliviano with the trigger price equal to $0.12$ and the strike price of $0.16$.

(a) $-0.0182$
(b) $-0.0024$
(c) $0$
(d) $0.0074$
(e) None of the above.
Solution: (a)

In our usual notation,

\[ d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{x(0)}{K_t} \right) + \left( r_D - r_F + \frac{\sigma^2}{2} \right) T \right] \]
\[ = \frac{1}{0.1 \sqrt{1}} \left[ \ln \left( \frac{0.14}{0.12} \right) + \left( 0.024 - 0.05 + \frac{0.1^2}{2} \right) \cdot 1 \right] = 1.3315, \]

\[ d_2 = d_1 - \sigma \sqrt{T} = 1.2315. \]

Using the standard normal table, we get

\[ N(d_1) = 0.9082, \quad N(d_2) = 0.8907. \]

So, the price of the gap option equals

\[ V_{GC}(0) = x(0)e^{-r_F T}N(d_1) - K_s e^{-r_D T}N(d_2) \]
\[ = 0.14e^{-0.05}(0.9082) - 0.16e^{-0.024}(0.8907) = -0.0182. \]
Problem 1.17. Assume that the stock price follows the Black-Scholes model. You are given the following information:

- The current stock price is $100.
- The mean rate of return on the stock is 0.15.
- The stock’s dividend yield is 0.01.
- The stock’s volatility is 0.35.
- The continuously-compounded, risk-free interest rate is 0.05.

Find the conditional expectation of the stock price at time−1 given that the time−1 stock price exceeds $80.

(a) $102.02
(b) $108.19
(c) $115.03
(d) $126.71
(e) None of the above.

Solution: (d)

Note that the given continuously-compounded, risk-free interest rate is not necessary to solve the problem!

In our usual notation, we have

\[ \hat{d}_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + \left( \alpha - \delta + \frac{\sigma^2}{2} \right) T \right] = \frac{1}{0.35 \sqrt{1}} \left[ \ln \left( \frac{100}{80} \right) + \left( 0.15 - 0.01 + \frac{0.35^2}{2} \right) \right] \]

\[ = 1.21255 \approx 1.21, \]

\[ \hat{d}_2 = \hat{d}_1 - \sigma \sqrt{T} = 1.21 - 0.35 = 0.86255 \approx 0.86. \]

Using the standard normal tables, we get

\[ N(\hat{d}_1) = 0.8869 \quad \text{and} \quad N(\hat{d}_2) = 0.8051. \]

Finally, our answer is

\[ \mathbb{E}[S(1) \mid S(1) > 80] = \frac{100e^{0.15-0.01\cdot(0.8869)}}{0.8051} = 126.7144. \]