Notes: This is a closed book and closed notes exam.
Time: 50 minutes

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FOR GRADER’S USE ONLY:

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1.1. **TRUE/FALSE QUESTIONS.** Please note your answers on the front page.

**Problem 1.1.** When a cap is paid in arrears, its price is equal to the price when it is paid in advance. *True or false?*

**Solution:** TRUE

**Problem 1.2.** The price of a cap can be calculated as the sum of the prices of caplets it consists of. *True or false?*

**Solution:** TRUE

**Problem 1.3.** In our usual notation, in a Black-Derman-Toy tree, we have that

\[ r_{uu} > r_u > r_{ud} \]

*True or false?*

**Solution:** FALSE

**Problem 1.4.** (2 points) Let \( X \) be a random variable whose variance is zero. Then \( X \) is constant with probability one. *True or false?*

**Solution:** TRUE

**Problem 1.5.** The interest-rate cap pays the difference between the realized interest rate in a period and the cap rate. *True or false?*

**Solution:** FALSE

The cap pays the excess above the cap rate if it is positive, not simply the difference between the two.
1.2. **FREE-RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they’re correct) are worth 0 points.

**Problem 1.6.** (15 points) The following interest-rate tree is used to model the evolution of effective interest rates over the following three-year period.

\[
\begin{align*}
    r_0 &= 0.035, & r_u &= 0.04, & r_{uu} &= 0.06, \\
    r_d &= 0.03, & r_{ud} &= 0.035, \\
    r_{dd} &= 0.025.
\end{align*}
\]

The interest rate is twice as likely to take an up step as it is to take a down step.

Consider a three-year, 3% cap purchased on a $100,000 loan with interest-only payments at the end of every year. What is the price of this cap?

**Solution:** The price of the cap is

\[
\begin{align*}
    100,000 \times \frac{1}{1.035} \times \left[ \frac{2}{3} \times \frac{0.04 - 0.03}{1.04} + \frac{4}{9} \times \frac{0.06 - 0.03}{(1.06)(1.04)} + \frac{2}{9} \times \frac{0.035 - 0.03}{1.035} \times \left( \frac{1}{1.04} + \frac{1}{1.03} \right) \right]
\end{align*}
\]

\[= 1,988.37\]
Problem 1.7. (10 points) Here is an (incomplete) Black-Derman-Toy tree
\[ r_0 = 0.035, \quad r_u = 0.04, \quad r_{uu} = 0.06, \]
\[ r_d = 0.025, \quad r_{ud} = 0.04. \]
Consider a call option on a bond with expiration date at time−1 which gives its owner the right to purchase a zero-coupon bond with redemption amount of $1 and two years left to maturity for $0.92. What is the time−0 price of this call option?

Solution: To complete the BDT tree, we calculate
\[ r_{dd} = \frac{r_{ud}^2}{r_{uu}} = \frac{0.0016}{0.06} = 0.02667. \]
At time−1, there are two possible bond prices according to the above BDT tree:
\[ P_u = \frac{1}{1.04} \times \frac{1}{2} \times \left[ \frac{1}{1.06} + \frac{1}{1.04} \right] = 0.915834, \]
\[ P_d = \frac{1}{1.025} \times \frac{1}{2} \times \left[ \frac{1}{1.02667} + \frac{1}{1.04} \right] = 0.944176. \]
The only non-zero payoff of our call option is at the down node and it equals
\[ 0.944176 - 0.92 = 0.024176. \]
So, the call’s price is
\[ \frac{1}{1.035} \times \frac{1}{2} \times 0.024176 = 0.0116793. \]
Problem 1.8. (20 points) Here is an (incomplete) Black-Derman-Toy tree

\[ r_u = 0.04, \quad r_{uu} = 0.06, \]
\[ r_d = 0.03, \quad r_{ud} = 0.04. \]

Compute the volatility in year-1 of the 3-year zero-coupon bond generated by the above tree.

Solution: To complete the BDT tree, we have

\[ r_{dd} = \frac{0.04^2}{0.06} = 0.0267. \]

The two possible time-1 bond prices are

\[ P_u = \frac{1}{1.04} \times \frac{1}{2} \times \left( \frac{1}{1.06} + \frac{1}{1.04} \right) = 0.915834, \]
\[ P_u = \frac{1}{1.03} \times \frac{1}{2} \times \left( \frac{1}{1.0267} + \frac{1}{1.04} \right) = 0.939579. \]

The two yields are

\[ y_u = \frac{1}{\sqrt{P_u}} - 1 = 0.044941, \]
\[ y_d = \frac{1}{\sqrt{P_d}} - 1 = 0.031652 \]

The required volatility is, hence,

\[ \kappa = \frac{1}{2} \ln \left( \frac{0.044941}{0.031652} \right) = 0.1752746 \]
**Problem 1.9.** (20 points) A discrete-time model is used to model both the price of a continuous-dividend-paying stock and the short-term (risk-free) interest rate. The dividend yield is given to be 0.02. Each period is one year.

The time−0 stock price is given to be $90 per share, while the one-year spot rate for zero-coupon bonds equals 0.04.

At time−1, there are two states of the world: *sunny* and *cloudy*. In the *sunny* state of the world, the stock price is $100 and the effective yearly interest rate is 0.03. In the *cloudy* state of the world, the stock price is $75 and the effective yearly interest rate is 0.05.

Consider a European put and a European call on the above stock whose strike is $90 and whose exercise date is at time−2. What is the difference between the two option prices?

**Solution:** The difference between the two options’ prices is

\[
V_C(0) - V_P(0) = F_{0,2}^P(S) - KP(0, 2) = S(0)e^{-2\delta} - KP(0, 2)
\]

The risk-neutral probability of the *sunny* state of the world is

\[
p^* = \frac{90(1.04)e^{-0.02} - 75}{100 - 75} = 0.6699.
\]

So,

\[
P(0, 2) = \frac{1}{1.04} \left[ \frac{1}{1.03} \times 0.6699 + \frac{1}{1.05} \times (1 - 0.6699) \right] = 0.9277.
\]

Finally, our answer is

\[
90e^{-0.04} - 90(0.9277) = 2.98146.
\]
1.3. **MULTIPLE CHOICE QUESTIONS.** Please, record your answers on the front page of this exam.

**Problem 1.10.** A binomial tree is used to model continuously compounded annual interest rates over the time period \([0, 2]\). The root interest rate is equal to 0.06. Thereafter, the interest rate can increase by 0.02 or decrease by 0.01. Assume that the up movement and the down movement in the tree are equally likely. What is the continuously compounded yield to maturity of a 2-year zero-coupon bond?

(a) 0.0566  
(b) 0.0598  
(c) 0.0605  
(d) 0.0624  
(e) None of the above.

**Solution: (d)**

The bond price dictated by the given interest-rate tree is

\[
P(0, 2) = e^{-0.06} \times \frac{1}{2} \left( e^{-0.08} + e^{-0.05} \right) = 0.8826.
\]

So, the yield rate \(y\) we are looking for satisfies

\[
e^{-2y} = P(0, 2) = 0.8826 \quad \Rightarrow \quad y = 0.0624.
\]

**Problem 1.11.** The one-year annual effective spot rate is given to be \(r_0(0, 1) = 0.03\). The annual effective implied forward rate for the time period \([1, 3]\) is \(r_0(1, 3) = 0.04\). What is the forward price for delivery in one year of a zero-coupon bond with two years left to maturity? Assume the redemption amount of $1.

(a) 0.9071  
(b) 0.9246  
(c) 0.9505  
(d) 0.9615  
(e) None of the above.

**Solution: (b)**

In our usual notation, the required forward price is

\[
F_{0.1}[P(1, 3)] = \frac{P(0, 3)}{P(0, 1)} = \frac{(1 + r_0(0, 1))^{-1}(1 + r_0(1, 3))^{-2}}{(1 + r_0(0, 1))^{-1}} = (1 + r_0(1, 3))^{-2} = 1.04^{-2} = 0.924556.
\]

**Problem 1.12.** A four-year, 7%-caplet is purchased on a $1 million dollar loan. The caplet’s payment is paid in advance. The realized floating annual effective interest rates are 0.08, 0.09, 0.07, 0.085.

What is the nominal amount of the payment of this caplet?

(a) 0  
(b) 13,824.90
(c) 14,018.70
(d) 15,000
(e) None of the above.

Solution: (b) \[
\frac{0.085 - 0.07}{1.085} = 13,824.90.
\]
Problem 1.13. A Black-Derman-Toy tree is calibrated so that $P(0, 1) = 0.925$ and $P(0, 2) = 0.84$. You are given that the yield volatility at the end of one year for bonds with one year remaining to expiration is 0.10.

What is the effective annual interest rate at the upper node in the BDT tree?

(a) 0.0901  
(b) 0.1003  
(c) 0.1040  
(d) 0.1114  
(e) None of the above.

Solution: (d)

We are given that the volatility of annual effective interest rates at the end of one year equals $\sigma_1 = 0.10$. So, the effective annual interest rate $r_u$ in the upper node in the tree must satisfy the following calibration equation

$$P(0, 2) = P(0, 1) \times \frac{1}{2} \times \left[ \frac{1}{1 + r_u} + \frac{1}{1 + r_u e^{-2\sigma_1}} \right].$$

Using the provided data, we get

$$\frac{2(0.84)}{0.925} = \frac{1}{1 + r_u} + \frac{1}{1 + r_u e^{-0.2}}.$$

We get $r_u = 0.1114$.

Problem 1.14. (5 points) It is observed that the bond prices for zero coupon bonds redeemable for $1 are, in our usual notation,

$P(0, 1) = 0.96$ and $P(0, 2) = 0.92$.

In the Black-Derman-Toy tree, the base rate parameter is set to equal $r_d = 0.035$. What is the interval into which the value of the volatility of effective interest rates in year-2 falls?

(a) [0, 0.1]  
(b) [0.1, 0.15]  
(c) [0.15, 0.25]  
(d) [0.25, 0.35]  
(e) None of the above.

Solution: (c)

We need to solve for the parameter $\sigma_1$ in the calibration equation. The calibration equation is

$$P(0, 2) = P(0, 1) \times \frac{1}{2} \left[ \frac{1}{1 + r_d} + \frac{1}{1 + r_d e^{2\sigma_1}} \right].$$

In the present problem, we have

$$0.92 = 0.96 \times \frac{1}{2} \left[ \frac{1}{1.035} + \frac{1}{1 + 0.035 e^{2\sigma_1}} \right].$$

So,

$$1.91667 = \frac{1}{1.035} + \frac{1}{1 + 0.035 e^{2\sigma_1}} \quad \Rightarrow \quad 1 + 0.035 e^{2\sigma_1} = 1.0521 \quad \Rightarrow \quad \sigma_1 = 0.198876.$$