NAME:

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
In-Term Exam I
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 80.
Time: 50 minutes

TRUE/FALSE

1.1 (2)	TRUE	FALSE
1.2 (2)	TRUE	FALSE
1.3 (2)	TRUE	FALSE
1.4 (2)	TRUE	FALSE
1.5 (2)	TRUE	FALSE
1.6 (2)	TRUE	FALSE
1.7 (2)	TRUE	FALSE
1.8 (2)	TRUE	FALSE

MULTIPLE CHOICE

1.11 (5)	a	b	c	d	e
1.12 (5)	a	b	c	d	e
1.13 (5)	a	b	c	d	e
1.14 (5)	a	b	c	d	e
1.15 (5)	a	b	c	d	e
1.16 (5)	a	b	c	d	e
1.17 (5)	a	b	c	d	e

FOR GRADER'S USE ONLY:

DEF'N	T/F	1.9	1.10	M.C.	Σ

1.1. **TRUE/FALSE QUESTIONS.** *Please note your answers on the front page.*

Problem 1.1. (2 points) Under the risk-neutral probability measure, every option on a particular stock has the continuously compounded, risk-free interest rate as its mean rate of return. *True or false?*

Solution: TRUE

Problem 1.2. The Black-Scholes option pricing formula can **always** be used for pricing American-type call options on non-dividend-paying assets. *True or false?*

Solution: TRUE

Problem 1.3. The Black-Scholes option pricing formula can as a rule only be used for pricing European-type options. *True or false?*

Solution: TRUE

Problem 1.4. In developing the log-normal stock price model, we assumed normally distributed realized returns. *True or false?*

Solution: TRUE

Problem 1.5. The probability density function of standard normal distribution is even, i.e., its graph is symmetric around the vertical axis. *True or false?*

Solution: TRUE

Problem 1.6. Let $S(T)$ be the lognormal stock price. Then, the expected payoff of a time$-T$, strike$-K$ European put option is at least $(K - \mathbb{E}[S(T)])_+$. *True or false?*

Solution: TRUE
This is a straightforward application of Jensen's inequality.

Problem 1.7. Assume the Black-Scholes framework for modeling the price of a continuous-dividend-paying stock. The mean stock price is a strictly increasing function of time regardless of the parameter values. *True or false?*

Solution: FALSE

Problem 1.8. (2 pts) In the setting of the Black-Scholes stock-price model, let $\mathbf{S} = \{S(t), t \geq 0\}$ denote the stock price whose drift is α and volatility is σ.

Then,

$$\mathbb{E}[S(t+h) \,|\, S(t)] = S(t), \text{ for every } t \geq 0 \text{ and } h > 0.$$

True or false?

Solution: FALSE

1.2. **FREE-RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.9. (20 points) Let $S(0) = \$120, K = \$100, \sigma = 0.3, r = 0$ and $\delta = 0.08$.

a. (10 pts) Let $V_C(0,T)$ denote the Black-Scholes European call price for the maturity T. Does the limit of $V_C(0,T)$ as $T \to \infty$ exist? If it does, what is it?

b. (8 pts) Now, set $r = 0.001$ and let $V_C(0,T,r)$ denote the Black-Scholes European call price for the maturity T. Again, how does $V_C(0,T,r)$ behave as $T \to \infty$?

c. (2 pts) Interpret in a sentence or two the differences, if any, between your answers to questions in a. and b.

Solution:

a. By the Black-Scholes pricing formula, the function $V_C(0,T)$ has the form

$$V_C(0,T) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) = S(0)e^{-\delta T}N(d_1) - K\,N(d_2),$$

where N denotes the distribution function of the unit normal distribution and

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(-\delta + \frac{1}{2}\sigma^2\right)T\right], \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

As $T \to \infty$, we have that

$$\begin{aligned} e^{-\delta T}N(d_1) &\leq e^{-\delta T} \to 0, \\ d_2 \to -\infty &\Rightarrow N(d_2) \to 0. \end{aligned}$$

Hence,

$$V_C(0,T) \to 0, \text{ as } T \to \infty.$$

b. In this case, the price of the call option reads as

$$V_C(0,T,r) = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2),$$

with

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T\right], \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

As $T \to \infty$, we have that

$$\begin{aligned} e^{-\delta T}N(d_1) &\leq e^{-\delta T} \to 0, \\ d_2 \to -\infty &\Rightarrow N(d_2) \to 0. \end{aligned}$$

Since the function N is bounded between 0 and 1, we see that as $T \to \infty$, $V_C(0,T,r) \to 0$.

c. Until the call option is exercised, the owner of the option can earn interest on the strike price which he/she can invest at the risk-free rate. However, in forfeiting the physical ownership of the asset, he/she also forfeits the possible dividend payments. It would be interesting to see what happens for $\delta = 0$.

Problem 1.10. (10 points) *Source: CAS, Exam 8, Spring 2000, Problem #31.*
You are considering the purchase of a three-month European put option on a stock with an announced dividend payment of \$1.50 in two months. You are given the following information:

- The strike price is \$50.
- The continuously compounded risk-free interest rate is 10%.
- The annual volatility of a three-month prepaid forward on the stock is 0.30.
- The stock follows the Black-Scholes framework.
- In our usual notation, $d_2 = -0.1086$.

Determine the current stock price.

Solution: Using the given value of d_2, we can solve for the prepaid forward price of the underlying stock.

$$\begin{aligned}-0.1086 = d_2 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{F^P_{0,T}(S)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T\right]\\ &= \frac{1}{0.3\sqrt{\frac{1}{4}}}\left[\ln\left(\frac{F^P_{0,T}(S)}{50}\right) + \left(0.1 - \frac{0.3^2}{2}\right)\frac{1}{4}\right].\end{aligned}$$

So,

$$-0.1086(0.15) = -0.01629 = \ln\left(\frac{F^P_{0,T}(S)}{50}\right) + (0.055)\frac{1}{4} = \ln\left(\frac{F^P_{0,T}(S)}{50}\right) + 0.01375.$$

Hence,

$$\frac{F^P_{0,T}(S)}{50} = e^{-0.03004} \quad \Rightarrow \quad F^P_{0,T}(S) = 50e^{-0.03004} = 48.52.$$

Finally,

$$S(0) = F^P_{0,T}(S) + 1.5e^{-0.10\left(\frac{1}{6}\right)} = 48.52 + 1.5e^{-0.10\left(\frac{1}{6}\right)} = 49.9955 \approx 50.$$

1.3. **MULTIPLE CHOICE QUESTIONS.** *Please note your asnwers on the front page.*

Problem 1.11. (5 points) Let $\mathbf{S} = \{S(t), t \geq 0\}$ denote the stock-price process following a lognormal distribution. The mean stock price at time-1 equals 120 and the median stock price 115. What is the probability that the time-1 stock price exceeds 100?

(a) About 0.6844
(b) About 0.7823
(c) About 0.8529
(d) About 0.8932
(e) None of the above.

Solution: **(a)**
Since $\mathbf{S} = \{S(t), t \geq 0\}$ is the stock-price process modeled by a geometric Brownian motion, the stock price at time-1 is lognormally distributed. In fact, using our usual parameters, we can rewrite it as

$$S(1) = S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)+\sigma Z(1)}.$$

Recall that the median of $S(1)$ equals $S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)}$. So, the required probability can be expressed as

$$\begin{aligned}\mathbb{P}[S(1) > 100] = \mathbb{P}[115e^{\sigma Z(1)} > 100] &= \mathbb{P}\left[Z(1) > \frac{1}{\sigma}\ln\left(\frac{100}{115}\right)\right] \\ &= \mathbb{P}\left[Z(1) < \frac{1}{\sigma}\ln\left(\frac{115}{100}\right)\right] = N\left(\frac{1}{\sigma}\ln\left(\frac{115}{100}\right)\right).\end{aligned}$$

Since the mean of $S(1)$ equals $S(0)e^{(\alpha-\delta)}$, we have

$$e^{\frac{1}{2}\sigma^2} = \frac{120}{115} \quad \Rightarrow \quad \sigma = \sqrt{2\ln(1.04348)} = 0.2918.$$

So, our final answer is

$$\mathbb{P}[S(1) > 100] = N(0.48) = 0.6844.$$

Problem 1.12. (5 pts) Assume the Black-Scholes framework for modeling the futures prices on oil. Let the futures price of oil for delivery in one year equal \$90.00. The volatility of this price is given to be 0.26.

Assume that the continuously compunded risk-free interest rate equals 0.05.

Consider an at-the-money European put option on the above futures contract with the exercise date in one year.

What is the Black-Scholes price $V_P(0)$ of this put option?

(a) \$6.62
(b) \$7.73
(c) \$8.85
(d) \$9.31
(e) None of the above.

Solution: (c)
We are given the one-year futures price on oil to be $F_{0,1} = 90$.

Since the option in question is given to be at-the-money, it is immediate that in our usual notation,

$$d_1 = \frac{1}{2}\sigma\sqrt{T} = \frac{0.25}{2} = 0.13, \quad d_2 = -d_1 = -0.13.$$

The price of the put option is

$$V_P(0) = F_{0,1}e^{-rT}(2N(-d_1) - 1) = 90e^{-0.05}(2 \times 0.5517 - 1) = 8.85.$$

Problem 1.13. **IFM Sample (Introductory) Problem #6.**
The following relates to one share of XYZ stock:

- The current price is 100.
- The forward price for delivery in one year is 105.
- An investor who decides to long the forward contract denotes by P the expected stock price in one year.

Determine which of the following statements about P is **TRUE**.

(A) $P < 100$
(B) $P = 100$
(C) $100 < P < 105$
(D) $P = 105$
(E) $P > 105$

Solution: **(e)**
Since the investor decided to long the forward contract, the payoff/profit will be

$$S(T) - 105$$

where $S(T)$ denotes the stock price on the delivery date T. The reason the investor chose to long the forward was the belief that the expected profit would be positive, i.e.,

$$\mathbb{E}[S(T)] = P > 105.$$

Problem 1.14. (5 pts) Which of the following gives the correct values for the delta and gamma of a single share of non-dividend-paying stock?

(a) $\Delta = 1, \Gamma = 1$
(b) $\Delta = 1, \Gamma = 0$
(c) $\Delta = 0, \Gamma = 1$
(d) $\Delta = 0, \Gamma = 0$
(e) None of the above.

Solution: **(b)**

Problem 1.15. (5 points) Let the current price of a continuous-dividend-paying stock be denoted by $S(0)$. We model the time$-T$ stock price as lognormal. The mean rate of return on the stock is 0.10, its dividend yield is 0.01, and its volatility is 0.20. The continuously-compounded, risk-free interest rate is 0.03. You invest in one share of stock at time-0. and simultaneously deposit an amount $\varphi S(0)$ in a savings account. Assume continuous and immediate reinvestment of all dividends in the same stock. What should the proportion φ be so that the VaR at the level 0.05 of your total wealth at time-1 equals today's stock price $S(0)$?

(a) $\varphi = 0.0573$
(b) $\varphi = 0.1966$
(c) $\varphi = 0.2139$
(d) $\varphi = 0.5$
(e) None of the above.

Solution: (c)
The total wealth at time-1 is equal to $e^{\delta}S(1) + \varphi S(0)e^{r}$. So, our condition on the VaR is

$$\mathbb{P}[e^{\delta}S(1) + \varphi S(0)e^{r} < S(0)] = 0.05.$$

The above is equivalent to

$$\mathbb{P}[S(0)e^{\alpha - \frac{\sigma^2}{2} + \sigma Z} + \varphi S(0)e^{r} < S(0)] = 0.05$$

with $Z \sim N(0,1)$. We have that the above is, in turn, equivalent to

$$\mathbb{P}[e^{\alpha - \frac{\sigma^2}{2} + \sigma Z} + \varphi e^{r} < 1] = 0.05.$$

The constant z^* such that $\mathbb{P}[Z < z^*] = 0.05$ equals -1.645. Therefore,

$$e^{\alpha - \frac{\sigma^2}{2} + \sigma(-1.645)} + \varphi e^{r} = 1.$$

So,

$$\varphi = e^{-r}\left(1 - e^{\alpha - \frac{\sigma^2}{2} + \sigma(-1.645)}\right) = e^{-0.03}\left(1 - e^{0.10 - \frac{(0.2)^2}{2} + 0.2(-1.645)}\right) = 0.2139.$$

Problem 1.16. (5 points) The current exchange rate of the Bolivian Boliviano to the USD is \$0.14 per Bolivian Boliviano. The volatility of the exchange rate is given to be 0.10. The USD continuously compounded, risk-free interest rate is $r_{\$} = 0.024$ while the Bolivian Boliviano continuously compounded, risk-free interest rate equals $r_B = 0.05$. Assume the Garman-Kohlhagen model is used for the exchange rate. Find the price of a one-year gap call option on the Bolivian Boliviano with the trigger price equal to \$0.12 and the strike price of \$0.16.

(a) -0.0182
(b) -0.0024
(c) 0
(d) 0.0074
(e) None of the above.

Solution: (a)
In our usual notation,

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{x(0)}{K_t}\right) + \left(r_D - r_F + \frac{\sigma^2}{2}\right)T\right] \\ &= \frac{1}{0.1\sqrt{1}}\left[\ln\left(\frac{0.14}{0.12}\right) + \left(0.024 - 0.05 + \frac{0.1^2}{2}\right)\cdot 1\right] = 1.3315, \\ d_2 &= d_1 - \sigma\sqrt{T} = 1.2315. \end{aligned}$$

Using the standard normal table, we get

$$N(d_1) = 0.9082, \quad N(d_2) = 0.8907.$$

So, the price of the gap option equals

$$\begin{aligned} V_{GC}(0) &= x(0)e^{-r_F T}N(d_1) - K_s e^{-r_D T}N(d_2) \\ &= 0.14e^{-0.05}(0.9082) - 0.16e^{-0.024}(0.8907) = -0.0182. \end{aligned}$$

Problem 1.17. Assume that the stock price follows the Black-Scholes model. You are given the following information:

- The current stock price is \$100.
- The mean rate of return on the stock is 0.15.
- The stock's dividend yield is 0.01.
- The stock's volatility is 0.35.
- The continuously-compounded, risk-free interest rate is 0.05.

Find the conditional expectation of the stock price at time-1 **given** that the time-1 stock price exceeds \$80.

(a) \$102.02
(b) \$108.19
(c) \$115.03
(d) \$126.71
(e) None of the above.

Solution: (d)
Note that the given continuously-compounded, risk-free interest rate is not necessary to solve the problem!

In our usual notation, we have

$$\begin{aligned}\hat{d}_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(\alpha - \delta + \frac{\sigma^2}{2}\right)T\right] = \frac{1}{0.35\sqrt{1}}\left[\ln\left(\frac{100}{80}\right) + \left(0.15 - 0.01 + \frac{0.35^2}{2}\right)\right] \\ &= 1.21255 \approx 1.21, \\ \hat{d}_2 &= \hat{d}_1 - \sigma\sqrt{T} = 1.21 - 0.35 = 0.86255 \approx 0.86.\end{aligned}$$

Using the standard normal tables, we get

$$N(\hat{d}_1) = 0.8869 \quad \text{and} \quad N(\hat{d}_2) = 0.8051.$$

Finally, our answer is

$$\mathbb{E}[S(1) \mid S(1) > 80] = \frac{100e^{0.15-0.01}(0.8869)}{0.8051} = 126.7144.$$