Name:

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
The Prerequisite In-Term Exam
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 75.
Time: 50 minutes

TRUE/FALSE

1.2 (2)	TRUE	FALSE
1.3 (2)	TRUE	FALSE
1.4 (2)	TRUE	FALSE
1.5 (2)	TRUE	FALSE
1.6 (2)	TRUE	FALSE
1.7 (2)	TRUE	FALSE
1.8 (2)	TRUE	FALSE

MULTIPLE CHOICE

1.12 (5)	a	b	c	d	e
1.13 (5)	a	b	c	d	e
1.14 (5)	a	b	c	d	e
1.15 (5)	a	b	c	d	e
1.16 (5)	a	b	c	d	e
1.17 (5)	a	b	c	d	e

FOR GRADER'S USE ONLY:

DEF	T/F	1.9	1.10	1.11	M.C.	Σ

1.1. DEFINITION.

Problem 1.1. (10 points)
Provide the definition of an **arbitrage portfolio**.

Solution: See your lecture notes from *M339D*.

1.2. **True/false questions.**

Problem 1.2. (2 points) It is never optimal to exercise an American call option on a non-dividend paying stock early. *True or false?*

Solution: TRUE

Problem 1.3. (2 points) A time$-T$ exchange call with underlying $\mathbf{S}$ and strike asset $\mathbf{Q}$ is always worth strictly more than an exchange put option with underlying $\mathbf{Q}$ and strike asset $\mathbf{S}$. *True or false?*

Solution: FALSE

Problem 1.4. (2 points) A bear spread is a long position with respect to the underlying asset.

Solution: FALSE

Problem 1.5. (2 points) In the binomial asset pricing model, the number of shares Δ of the underlying asset in the replicating portfolio for a **put** option is always positive.

Solution: FALSE

Problem 1.6. (2 points) If the random variable X has the distribution function F_X, then the distribution function of the random variable $Y = |X|$ equals

$$F_Y(y) = 2F_X(y).$$

True or false?

Solution: FALSE

Problem 1.7. (2 points) Let $X_1, \dots, X_n$ be random variables with finite expectations and let $\alpha_1, \dots, \alpha_n$ be constants. Then, we always have that

$$\mathbb{E}[\alpha_1 X_1 + \cdots + \alpha_n X_n] = \sum_{i=1}^{n} \alpha_i \mathbb{E}[X_i]\,.$$

True or false?

Solution: TRUE

Problem 1.8. (2 points) A linear combination of two normally distributed random variables is always also normally distributed. Assume that a constant is also considered normally distributed with variance zero. *True or false?*

Solution: TRUE

1.3. **Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.9. (5 points) An investor short sells one share of a non-dividend-paying stock and buys an at-the-money, T-year, European call option on this stock. The call premium is denoted by $V_C(0)$. Assume that there are no transaction costs. The continuously compounded, risk-free interest rate is denoted by r. Let the argument s represent the stock price at time T.

(i) (3 points) Determine an algebraic expression for the investor's profit at expiration T in terms of $V_C(0), r, T$ and the strike K.

(ii) (2 points) In particular, how does the expression you obtained in (i) simplify if the call is in-the-money on the exercise date?

Solution:

$$-s + (s-K)_+ + (S(0) - V_C(0))e^{rT} = -s + (s-K)_+ + (K - V_C(0))e^{rT}.$$

For the option to be in-the-money at expiration, we must have $s < K$. So, the profit simplifies to

$$-s + (s-K) + (K - V_C(0))e^{rT} = -K + (K - V_C(0))e^{rT}.$$

Problem 1.10. (10 pts) For a two-period binomial model, you are given that:

(1) each period is one year;
(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;
(3) $u = 1.2$, with u as in the standard notation for the binomial model;
(4) $d = 0.8$, with d as in the standard notation for the binomial model;
(5) the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **chooser** option such that its owner can decide after one year whether the option becomes a put or a call option with exercise date at time-2 and strike equal to \$20.

Find the price of the chooser option.

Solution: The risk-neutral probability is

$$p^* = \frac{e^{0.04} - 0.8}{1.2 - 0.8} = 0.6020.$$

When one constructs the two-period binomial tree, one gets

$$S_u = 24,\ S_d = 16,$$
$$S_{uu} = 28.80,\ S_{ud} = S_{dd} = 19.2,\ S_{dd} = 12.8$$

The call will be worth more than the put in the *up* node while the put will be worth more than the call in the *down* node. This means that the chooser option's owner will choose the call in the *up* node and will choose the put in the *down* node.

The possible payoffs of the call at the end of the second period are

$$V_{uu} = 8.80 \quad \text{and} \quad V_{ud} = 0.$$

So, taking the discounted expected value at the *up* node of the payoff with respect to the risk-neutral probability, we get that the price of this call (and, hence, the price of the chooser option) at the *up* node equals

$$V_u^{CH} = e^{-0.04} \times 8.80 \times 0.602 = 5.0899.$$

The possible payoffs of the put at the end of the second period are

$$V_{ud} = 0.80 \quad \text{and} \quad V_{dd} = 7.20.$$

So, taking the discounted expected value at the *down* node of the payoff with respect to the risk-neutral probability, we get that the price of this put (and, hence, the price of the chooser option) at the *down* node equals

$$V_d^{CH} = e^{-0.04}\,[0.80 \times 0.602 + 7.20 \times 0.398] = 3.21595.$$

Finally, the time-0 price of the chooser option equals

$$V_{CH}(0) = e^{-0.04}\,[5.0899 \times 0.602 + 3.21595 \times 0.398] = 4.1737. \tag{1.1}$$

Problem 1.11. (10 pts) Let Z be a standard normal random variable. Find the following probabilities:

i. $\mathbb{P}[-1.33 < Z \leq 0.24]$
ii. $\mathbb{P}[0.49 < |Z|]$
iii. $\mathbb{P}[Z^4 < 0.0256]$
iv. $\mathbb{P}[e^{2Z} < 2.25]$
v. $\mathbb{P}\left[\frac{1}{Z} < 2\right]$

Solution:

i.
$$\begin{aligned}\mathbb{P}[-1.33 < Z \leq 0.24] &= \mathbb{P}[Z \leq 0.24] - \mathbb{P}[Z \leq -1.33] = \mathbb{P}[Z \leq 0.24] - (1 - \mathbb{P}[Z \leq 1.33]) \\ &= 0.5948 - 1 + 0.9082 = 0.503\end{aligned}$$

ii.
$$\begin{aligned}\mathbb{P}[0.49 < |Z|] &= \mathbb{P}[Z < -0.49] + \mathbb{P}[0.49 < Z] = 2\mathbb{P}[Z > 0.49] \\ &= 2(1 - \mathbb{P}[Z \leq 0.49]) = 2(1 - 0.6879) = 0.6242\end{aligned}$$

iii.
$$\begin{aligned}\mathbb{P}[Z^4 < 0.0256] &= \mathbb{P}[|Z| < \sqrt[4]{0.0256}] = \mathbb{P}[|Z| < 0.4] = \mathbb{P}[Z < 0.4] - \mathbb{P}[Z < -0.4] \\ &= 2\mathbb{P}[Z < 0.4] - 1 = 2(0.6554) - 1 = 0.3108\end{aligned}$$

iv.
$$\mathbb{P}[e^{2Z} < 2.25] = \mathbb{P}[2Z < \ln(2.25)] = \mathbb{P}[Z < 0.5\ln(2.25)] \approx \mathbb{P}[Z \leq 0.41] = 0.6591$$

v.
$$\begin{aligned}\mathbb{P}\left[\frac{1}{Z} < 2\right] &= \mathbb{P}[\tfrac{1}{Z} < 0] + \mathbb{P}[0 < \tfrac{1}{Z} < 2] \\ &= \mathbb{P}[Z < 0] + \mathbb{P}[Z > 0.5] = 0.5 + (1 - \mathbb{P}[Z \leq 0.5]) = 0.5 + (1 - 0.6915) = 0.8085.\end{aligned}$$

1.4. **MULTIPLE CHOICE QUESTIONS.** *Please note your answers on the front page.*

Problem 1.12. (5 points) A discrete-dividend-paying stock sells today for \$100 per share. The continuously compounded, risk-free interest rate is 0.04. The first dividend will be paid at in three months in the amount of \$2. The remaining dividends will be equal to \$1 and continue to be paid out quarterly. What is the **forward price** of this stock for delivery in seven months?

(a) \$73.02
(b) \$97.04
(c) \$99.33
(d) \$100
(e) None of the above.

Solution: The correct answer is **(c)**.

$$F^P_{0,7/12}(S) = 100 - 2e^{-0.01} - 1e^{-0.02} = 97.0397.$$

So,

$$F_{0,7/12}(S) = 97.0397e^{0.04(7/12)} = 99.3306. \tag{1.2}$$

Problem 1.13. (5 points) The random vector (X_1, X_2) is jointly normal. Its marginal distributions are:

$$X_1 \sim N(\text{mean} = 0, \text{variance} = 4), \quad X_2 \sim N(\text{mean} = 1, \text{variance} = 1).$$

The correlation coefficient is given to be

$$corr[X_1, X_2] = -0.2.$$

What is the variance of the random variable $X = 3X_1 - 2X_2$?

(a) 32.8
(b) 47.2
(c) 54.4
(d) 58.2
(e) None of the above.

Solution: **(e)**

The variance of X is

$$\begin{aligned} Var[X] &= 9Var[X_1] + 4Var[X_2] - 2(3)(2)Cov[X_1, X_2] \\ &= 9(4) + 4(1) + 12(2)(1)(0.2) = 44.8. \end{aligned}$$

Problem 1.14. In our usual notation, which of the parameter choices below creates a binomial model with an arbitrage opportunity?

(a) $u = 1.18, \quad d = 0.87, \quad r = 0.05, \quad \delta = 0, \quad h = 1/4$
(b) $u = 1.23, \quad d = 0.80, \quad r = 0.05, \quad \delta = 0.06, \quad h = 1/2$
(c) $u = 1.08, \quad d = 1, \quad r = 0.05, \quad \delta = 0.04, \quad h = 1$
(d) $u = 1.28, \quad d = 0.78, \quad r = \delta, \quad h = 2$
(e) None of the above.

Solution: (e)

Problem 1.15. The current price of a non-dividend-paying stock is \$50 per share. You observe that the price of a three-month, at-the-money American call option on this stock equals \$3.50.

The continuously compounded risk-free interest rate is 0.04.

Find the premium of the European three-month, at-the-money put option on the same underlying asset.

(a) About \$3
(b) About \$3.50
(c) About \$4
(d) About \$5.46
(e) None of the above.

Solution: (a)
Recall that the price of an American call on a non-dividend-paying stock equals the price of the otherwise identical European call option. So, put-call parity yields

$$V_P(0) = V_C(0) + Ke^{-rT} - S(0) = 3.50 - 50(e^{-0.01} - 1) = 3.0025.$$

Problem 1.16. The current stock price is observed to be \$100 per share. The stock is projected to pay dividends continuously at the rate proportional to its price with the dividend yield of 0.03. The stock's volatility is given to be 0.23. You model the evolution of the stock price using a two-period forward binomial tree with each period of length one year.

The continuously compounded risk-free interest rate is given to be 0.04.

What is the price of a two-year, \$101-strike **American** put option on the above stock consistent with the above stock-price tree?

(a) About \$6.62
(b) About \$8.34
(c) About \$8.83
(d) About \$11.11
(e) None of the above.

Solution: (d)
The up and down factors in the forward tree are

$$u = e^{0.01+0.23} = 1.2712, \quad \text{andd} = e^{-0.22} = 0.8025.$$

The risk-neutral probability equals

$$p^* = \frac{1}{1 + e^{0.23}} = 0.4428.$$

Of course, in the exam, you would not necessarily populate the entire stock-price tree, since you would want to work efficiently and only consider the nodes you need for pricing.

Evidently, the option produces a positive payoff only in the down-down node; the value of this payoff is $V_{dd} = 101 - 64.40 = 36.60$.

The continuation value at the down node is, hence,

$$CV_d = e^{-0.04} \times (1 - 0.4428) \times 36.60 = 15.57.$$

On the other hand, the value of immediate exercise at the down node equals $IE_d = 101 - 80.25 = 20.75$. So, it is optimal to exercise the American put in the down node and the value of the American put equals $V_d^P = 20.75$.

In the up node, both the continuation value and the immediate exercise value are zero. So, the initial price of the American put is

$$V_P(0) = e^{-0.04} \times (1 - 0.4428) \times 20.75 = 11.11$$

Problem 1.17. Consider a non-dividend paying stock whose current price is $95 per share. You model the evolution of this stock price over the following year using a one-period binomial tree under the assumption that the stock price can be either $120, or $75 in one year.

The continuously compounded risk-free interest rate is 0.06.

Consider a $100-strike, one-year European **straddle** on the above stock. What is the straddle's price consistent with the above stock-price model?

(a) About $10
(b) About $10.83
(c) About $15.45
(d) About $20.84
(e) None of the above.

Solution: (d)
The risk-neutral probability of an up movement is

$$p^* = \frac{95e^{0.06} - 75}{120 - 75} = 0.575.$$

So, the price of our straddle is

$$V(0) = e^{-0.06}[0.575 \times (120 - 100) + (1 - 0.575) * (100 - 75)] = 20.8366.$$