M39W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
Sample In-Term Exam II
Instructor: Milica Ćudina

Notes: This is a closed book and closed notes exam.
Time: 50 minutes

MULTIPLE CHOICE

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FOR GRADER’S USE ONLY:

T/F | 1. | 2. | M.C. | Σ
2.1. **TRUE/FALSE QUESTIONS.** Please note your answers on the front page.

**Problem 2.1.** (2 points) Under the risk-neutral probability measure, every option on a particular stock has the continuously compounded, risk-free interest rate as its mean rate of return. True or false?

**Solution:** TRUE

**Problem 2.2.** The Black-Scholes option pricing formula can always be used for pricing American-type call options on non-dividend-paying assets. True or false?

**Solution:** TRUE

**Problem 2.3.** Continuously compounded returns of stocks are multiplicative.

**Solution:** FALSE

**Problem 2.4.** Let $X$ be a strictly positive random variable, then

\[ \mathbb{E}[X] = |\mathbb{E}[X]|. \]

**Solution:** TRUE

**Problem 2.5.** In the lognormal stock-price model, the mean time$^{-t}$ stock price is increasing as a function of $t$.

**Solution:** FALSE

**Problem 2.6.** Let the stock price be modeled by a lognormal distribution. Then, the median stock price always exceeds the mean stock price. True or false?

**Solution:** FALSE

**Problem 2.7.** Let the stock price be modeled by a lognormal distribution. Then, the expected payoff of a European put option with exercise date $T$ and strike $K$ greater than or equal to $\max(0, K - \mathbb{E}[S(T)])$. True or false?

**Solution:** TRUE

**Problem 2.8.** In developing the log-normal stock price model, we assumed independent, identically distributed realized returns.

True or false?

**Solution:** TRUE
Problem 2.9. (2 pts)
In the setting of the Black-Scholes stock-price model, let \( S = \{S(t), t \geq 0\} \) denote the stock price whose drift is \( \alpha \) and volatility is \( \sigma \).

Then,
\[
\mathbb{E}[S(t + h) | S(t)] = S(t), \text{ for every } t \geq 0 \text{ and } h > 0.
\]

Solution: FALSE

Problem 2.10. The product of log-normal random variables is normal.

Solution: FALSE

2.2. FREE-RESPONSE PROBLEMS. Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they’re correct) are worth 0 points.

Problem 2.11. (15 points)
Consider a non-dividend-paying stock whose price is modeled using the lognormal distribution. Suppose that the current stock price equals $100 and that its volatility is given to be 0.2.

The continuously compounded, risk-free interest rate is assumed to equal 0.04.

Consider a derivative security which entitles its owner to obtain a European call option on the above stock six months from today, i.e., at time \( t^* = \frac{1}{2} \). The call option is to be half a year to expiration at time of delivery and have the strike equal to 105\% of the time–\( t^* \) price of the underlying asset. This contract is called a forward start option.

What is the price of the forward start option?

Solution:
At time \( t^* = \frac{1}{2} \), the Black-Scholes price of the call option to be delivered equals
\[
V_C(t^*) = S(t^*)N(d_1) - 1.05S(t^*)e^{-r(T-t^*)}N(d_2)
= S(t^*)(N(d_1) - 1.05e^{-0.02}N(d_2))
\]
with
\[
\begin{align*}
d_1 &= \frac{1}{0.2\sqrt{0.5}} \left[ -\ln(1.05) + \left( 0.04 + \frac{0.2^2}{2} \right) \times \frac{1}{2} \right] = -0.13, \\
d_2 &= d_1 - \sigma\sqrt{T-t^*} = -0.27.
\end{align*}
\]
Hence,
\[
V_C(t^*) = S(t^*)(0.4483 - 1.05e^{-0.02} \times 0.3936) = S(t^*)(0.0432).
\]
So, one would need to buy 0.0432 shares of stock to be able to buy the call option in question at time-\( t^* \). This amount of shares costs $4.32.
Problem 2.12. (10 points) Let the stock prices be modeled using the lognormal distribution. The mean stock price at time $-1$ equals 120 and the median stock price 115. What is the probability that the time $-1$ stock price exceeds 100?

Solution: The stock price at time $-1$ is lognormally distributed. In fact, using our usual parameters, we can rewrite it as

$$S(1) = S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)+\sigma Z(1)}.$$ 

Recall that the median of $S(1)$ equals $S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)}$. So, the required probability can be expressed as

$$\mathbb{P}[S(1) > 100] = \mathbb{P}[115e^{\sigma Z(1)} > 100] = \mathbb{P} \left[ Z(1) > \frac{1}{\sigma} \ln \left( \frac{100}{115} \right) \right]$$

$$= \mathbb{P} \left[ Z(1) < \frac{1}{\sigma} \ln \left( \frac{115}{100} \right) \right] = N \left( \frac{1}{\sigma} \ln \left( \frac{115}{100} \right) \right).$$

Since the mean of $S(1)$ equals $S(0)e^{(\alpha-\delta)}$, we have

$$e^{\frac{1}{2}\sigma^2} = \frac{120}{115} \implies \sigma = \sqrt{2\ln(1.04348)} = 0.2918.$$ 

So, our final answer is

$$\mathbb{P}[S(1) > 100] = N(0.48) = 0.6844.$$ 

2.3. MULTIPLE CHOICE QUESTIONS.

Problem 2.13. (5 points) The current price of a continuous-dividend-paying stock is given to be $92. The stock’s volatility is 0.35 and its dividend yield is 0.02. The continuously compounded risk-free interest rate is 0.05. Consider a $90-strike European call option on the above stock with exercise date in a quarter-year.

What is the Black-Scholes price of this call option?

(a) 5.05  
(b) 7.66  
(c) 7.71  
(d) 7.89  
(e) None of the above.

Solution: (d)

$$d_1 = 0.26, d_2 = 0.08.$$ 

So,

$$V_C(0) = 92e^{-0.02/4} \times 0.6026 - 90e^{-0.05/4} \times 0.5319 \approx 7.89.$$
Problem 2.14. A continuous-dividend-paying stock has a dividend yield of 0.02. The time—\( t \) realized (rate of) return is modeled as
\[
R(0, t) \sim N(\text{mean} = 0.03t, \text{variance} = 0.09t)
\]
Find the probability that the time—4 stock price exceeds today’s stock price.
(a) 0.4920
(b) 0.4960
(c) 0.5040
(d) 0.5080
(e) None of the above.

Solution: (e)
We need to find
\[
P[S(4) > S(0)]
\]
with
\[
S(t) = S(0)e^{R(0,t)}.
\]
Since \( R(0, t) \) follows the normal distribution with the above parameters, we have
\[
P[S(4) > S(0)] = P[S(0)e^{R(0,4)} > S(0)] = P[R(0, 4) > 0]
\]
\[
= 1 - N \left( \frac{-0.03 \times 4}{0.3 \times 2} \right) = N(0.2) = 0.5793.
\]
Problem 2.15. (5 pts) Assume the Black-Scholes framework. Let the current price of a non-dividend-paying stock be equal to $S(0) = 95$ and let its volatility be equal to 0.35. Consider a European call on that stock with strike 100 and exercise date in 9 months. Let the risk-free continuously compounded interest rate be 6% per annum.

Denote the price of the call by $V_C(0)$. Then,

(a) $V_C(0) < 5.20$
(b) $5.20 \leq V_C(0) < 7.69$
(c) $7.69 \leq V_C(0) < 9.04$
(d) $9.04 \leq V_C(0) < 11.25$
(e) None of the above.

Solution: (d)
Using the Black-Scholes formula one gets the price of about 11.06.

Problem 2.16. (5 pts) Assume the Black-Scholes framework. Let the current price of a share of stock be equal to $S(0) = 80$, let its volatility be $\sigma = 0.3$, and let $\delta = 0$ (in our usual notation).

Consider a gap option with expiration date $T = 1$ year such that its payoff is $S(T) - 90$ if $S(T) > 100$.

You are given that the continuously compounded risk-free interest rate equals $r = 0.05$ per annum.

Let $V_{GC}(0)$ denote the price of the above gap option. Then,

(a) $V_{GC}(0) < 3.20$
(b) $3.20 \leq V_{GC}(0) < 5.69$
(c) $5.69 \leq V_{GC}(0) < 7.04$
(d) $7.04 \leq V_{GC}(0) < 11.25$
(e) None of the above.

Solution: (c)
In our usual notation, the Black-Scholes formula for the price of a gap call option reads as

$$V_{GC}(0) = S(0)^{-\delta T}N(d_1) - K_1 e^{-r T}N(d_2)$$

where

$$d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln(S(0)/K_2) + (r - \delta + \frac{1}{2} \sigma^2)T \right],$$
$$d_2 = d_1 - \sigma \sqrt{T}.$$

In the present problem,

$$d_1 = \frac{1}{0.3} \left[ \ln(80/100) + (0.05 + \frac{0.09}{2}) \right] \approx -0.43,$$
$$d_2 = -0.73.$$

So, the price equals

$$V_{GC}(0) = 80N(-0.43) - 90e^{-0.05}N(-0.73) = 80 \cdot (1 - 0.6664) - 90e^{-0.05} \cdot (1 - 0.7673) = 6.7664.$$
Problem 2.17. Assume the Black-Scholes setting. Assume $S(0) = \$63.75, \sigma = 0.20, r = 0.055$. The stock pays no dividend and the option expires in 50 days (simplify the number of days in a year to 360). What is the price of a $60$-strike European put?

(a) 0.66
(b) 0.55
(c) 0.44
(d) 0.37
(e) None of the above.

Solution: (d) In our usual notation, the price is

$$V_P(0) = Ke^{-rT} N(-d_2) - S(0) N(-d_1)$$

with

$$d_1 = \frac{1}{0.2\sqrt{5/36}} \left( \ln \left( \frac{63.75}{60} \right) + (0.055 + \frac{1}{2} \cdot 0.2^2) \left( \frac{5}{36} \right) \right) = 0.95,$n$$

$$d_2 = d_1 - 0.25\sqrt{0.125} = 0.88.$$n

So,

$$V_P(0) = 60e^{-0.055\cdot\frac{5}{36}} (1 - 0.8106) - 63.75 \cdot (1 - 0.8289) = 0.37.$$n

Problem 2.18. Assume the Black-Scholes setting. Assume $S(0) = \$28.50, \sigma = 0.32, r = 0.04$. The stock pays a $1.0\%$ continuous dividend and the option expires in 110 days (simplify the number of days in a year to 360). What is the price of a $30$-strike put?

(a) 2.75
(b) 2.10
(c) 1.80
(d) 1.20
(e) None of the above.

Solution: (a) In our usual notation, the price is

$$V_P(0) = Ke^{-rT} N(-d_2) - S(0)e^{-\delta T} N(-d_1)$$

with

$$d_1 = -0.15, \quad d_2 = -0.33.$$n

So, $V_P(0) = 2.75$. 
Problem 2.19. (5 points) Consider a non-dividend-paying stock whose price \( S = \{ S(t), t \geq 0 \} \) is modeled using a geometric Brownian motion. Suppose that the current stock price equals $100 and that its volatility is given to be 0.25.

The continuously compounded, risk-free interest rate is assumed to equal 0.04.

Consider a derivative security which entitles its owner to obtain a European call option on the above stock six months from today, i.e., at time \( t^* = 1/2 \). The call option is to be 3-month to expiration at time of delivery and have the strike equal to 105% of the time–\( t^* \) price of the underlying asset. This contract is called a forward start option.

What is the price of the forward start option?

(a) 3.15
(b) 8.65
(c) 10.51
(d) 13.55
(e) None of the above.

Solution: (a)

At time \( t^* \), the required Black-Scholes price of the call option equals

\[
V_C(t^*) = S(t^*)N(d_1) - 1.05S(t^*)e^{-r(T-t^*)}N(d_2)
\]

\[
= S(t^*)(N(d_1) - 1.05e^{-0.01N(d_2)})
\]

with

\[
d_1 = \frac{1}{0.125} \left[ -\ln(1.05) + \left( 0.04 + \frac{0.25^2}{2} \right) \times \frac{1}{4} \right] = -0.2478,
\]

\[
d_2 = d_1 - \sigma \sqrt{T - t^*} = -0.3728.
\]

Hence,

\[
V_C(t^*) = S(t^*)(0.4013 - 1.05e^{-0.01 \times 0.3557}) =
\]

So, one would need to buy 0.0315 shares of stock to be able to buy the call option in question at time–\( t^* \). This amount of shares costs $3.15.

Problem 2.20. Assume the Black-Scholes setting.

Today’s price of a non-dividend paying stock is $65, and its volatility is 0.20.

The continuously compounded risk-free interest rate is 0.055.

What is the price of a three-month, $60-strike European put option on the above stock?

(a) 0.66
(b) 0.59
(c) 0.44
(d) 0.37
(e) None of the above.

Solution: (b) or (e)

In our usual notation, the price is

\[
V_P(0) = Ke^{-rT}N(-d_2) - S(0)N(-d_1)
\]
with
\[ d_1 = \frac{1}{0.2\sqrt{0.25}} \left( \ln \left( \frac{65}{60} \right) + (0.055 + \frac{1}{2} \cdot 0.2^2) \left( \frac{1}{4} \right) \right) = 10(\ln(65/60) + (0.075)(0.25)) = 0.99, \]
\[ d_2 = d_1 - 0.2\sqrt{0.25} = 0.89. \]
So,
\[ V_P(0) = Ke^{-rT}N(-d_2) - S(0)e^{-\delta T}N(-d_1) \]

**Problem 2.21.** Assume the Black-Scholes setting.

Today’s stock price is observed to be \( S(0) = $30 \) per share. Its dividend yield is given to be 0.01 and its volatility equals 0.30.

The continuously compounded risk-free interest rate is \( r = 0.04 \).

What is the price of a half-year, $30-strike put?

(a) 2.75 
(b) 2.38 
(c) 1.80 
(d) 1.20 
(e) None of the above.

**Solution:** (b)

In our usual notation, the price is
\[ V_P(0) = Ke^{-rT} N(-d_2) - S(0)e^{-\delta T} N(-d_1) \]

with
\[ d_1 = \frac{1}{0.30\sqrt{0.5/2}} \left[ \ln(30/30) + (0.04 - 0.01 + 0.045)(0.5) \right] = \frac{0.075}{0.3} \cdot \sqrt{0.5} = 0.18, \]
\[ d_2 = d_1 - 0.3\sqrt{0.5} = -0.04. \]

So, our final answer is
\[ V_P(0) = Ke^{-rT} N(-d_2) - S(0)e^{-\delta T} N(-d_1) \]
= \[ 30 \left[ e^{-0.04/2}(0.516) - e^{-0.01/2}(1 - 0.5714) \right] \]
= 2.3796.