"Confidence" intervals

$s^u$ and $s^l$ are constants such that

$\Pr [ S(T) > s^u ] = \frac{1}{2}$

and

$\Pr [ S(T) < s^l ] = \frac{1}{2}$
50. Assume the Black-Scholes framework. i.e., LogNormal stock prices.

You are given the following information for a stock that pays dividends continuously at a rate proportional to its price.

(i) The current stock price is 0.25. \( S(0) = 0.25 \)
(ii) The stock’s volatility is 0.35. \( \sigma = 0.35 \)
(iii) The continuously compounded expected rate of stock-price appreciation is 15%. \( \alpha - \delta = 0.15 \)

Calculate the upper limit of the 90% lognormal confidence interval for the price of the stock in 6 months.

Find \( s^u \) such that:

\[
\Pr[S(\frac{T}{2}) > s^u] = 0.05
\]

\[
s^u = 0.25e^{(0.15 - (0.35)^2)\cdot\frac{1}{2}} + 0.35\cdot\sqrt{\frac{1}{2}} \cdot 1.65
\]

\[
s^u = 0.393 \
\Rightarrow \text{(A)}
\]
LogNormal Stock-price Model

Inspiration: ASSET-OR-NOTHING CALL

\[ \Rightarrow \text{Payoff: } V_{Ac}(T) = S(T) \mathbb{I}_{[S(T) \geq K]} \]

Q: What is the expected payoff of the asset call?

\[ E[V_{Ac}(T)] = E[S(T) \mathbb{I}_{[S(T) \geq K]}] \]

- \( S(T) = S(0) e^{(\alpha - \delta - \frac{\sigma^2}{2}) T + \sigma \sqrt{T} \cdot Z} \) \( w/ \ Z \sim N(0,1) \)

- Review:
  \[
  \begin{align*}
  & \cdot X \ldots \text{a continuous random variable} \\
  & \quad w/ \ f_X \text{ as its probability density function} \\
  & \cdot h \ldots \text{a (continuous) real function defined on the support of } X \\
  & \quad E[h(X)] := \int_{-\infty}^{\infty} h(x) \cdot f_X(x) \, dx \quad \text{if the integral exists!}
  \end{align*}
  \]

- \( X \leftrightarrow Z \) : \( f_X \leftrightarrow \varphi \)
  \( (\alpha - \delta - \frac{\sigma^2}{2}) T + \sigma \sqrt{T} \cdot x \)

- \( h \leftrightarrow h(x) = S(0) e^{(\alpha - \delta - \frac{\sigma^2}{2}) T + \sigma \sqrt{T} \cdot x} \cdot \mathbb{I}_{[x \geq \hat{d}_2]} \)
After a bit of work:

\[
E \left[ S(T) \mathbb{I}_{[S(T) \geq K]} \right] = S(0) e^{(\alpha - \delta) t} \cdot N(\hat{\sigma}_4)
\]

w/ \( \hat{\sigma}_4 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + (\alpha - \delta + \frac{\sigma^2}{2}) \cdot T \right] \)

Asset-or-nothing put

\[
V_{AP}(T) = S(T) \mathbb{I}_{[S(T) < K]}
\]

\[
\Rightarrow E[V_{AP}(T)] = E[S(T) \mathbb{I}_{[S(T) < K]}] = E[S(T)] - E[S(T) \mathbb{I}_{[S(T) \geq K]}]
\]

\[
= E[S(T)] - E[S(T)] = S(0) e^{(\alpha - \delta) t} \cdot N(\hat{\sigma}_4)
\]

\[
= S(0) e^{(\alpha - \delta) t} (1 - N(\hat{\sigma}_4))
\]

\[
= S(0) e^{(\alpha - \delta) t} N(-\hat{\sigma}_4)
\]
Conditional Expectation
\[ E[S(T) \mid S(T) > K] = ? \]

By definition:
\[ E[S(T) \mid S(T) > K] = \frac{E[S(T)1_{[S(T) > K]}]}{P[S(T) > K]} \]
\[ = \frac{S(0)e^{(\alpha - \delta)T}N(\hat{d}_1)}{N(\hat{d}_2)} \]
\[ w/ \hat{d}_2 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + (\alpha - \delta - \frac{\sigma^2}{2})T \right] \]

Note: \( \hat{d}_2 = \hat{d}_1 - \sigma \sqrt{T} \)

\[ E[S(T) \mid S(T) < K] = \frac{E[S(T)1_{[S(T) < K]}]}{P[S(T) < K]} \]
\[ = \frac{S(0)e^{(\alpha - \delta)T}N(-\hat{d}_1)}{N(-\hat{d}_2)} \]

because \( P[S(T) < K] = 1 - P[S(T) > K] = 1 - N(\hat{d}_2) = N(-\hat{d}_2) \)
Problem. A non-dividend-paying stock is valued at $100 per share. The annual mean rate of return is 12%. The volatility is 30%. Assuming the log-normal stock-price model, find \( E [S(2) \mid S(2) > 95] = ? \)