On Market-Making and Delta-Hedging

1. Market Makers

2. Market-Making and Bond-Pricing

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What to market makers do?

- Provide **immediacy** by standing ready to sell to buyers (at ask price) and to buy from sellers (at bid price)
- Generate **inventory** as needed by short-selling
- **Profit** by charging the bid-ask spread
- Their position is determined by the order flow from customers
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Market Maker Risk

- Market makers attempt to hedge in order to avoid the risk from their arbitrary positions due to customer orders (see Table 13.1 in the textbook)
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Delta and Gamma as measures of exposure

- Suppose that Delta is 0.5824, when S = $40 (same as in Table 13.1 and Figure 13.1)
- A $0.75 increase in stock price would be **expected** to increase option value by $0.4368 (incerase in price $\times$ Delta = $0.75 x 0.5824)
- The **actual** increase in the options value is higher: $0.4548
- This is because the Delta increases as stock price increases. Using the smaller Delta at the lower stock price **understates** the the actual change
- Similarly, using the original Delta **overstates** the change in the option value as a response to a stock price **decline**
- Using Gamma in addition to Delta improves the approximation of the option value change (Since Gamma measures the change in Delta as the stock price varies - it's like adding another term in the Taylor expansion)

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Outline

- The Black model is a version of the Black-Scholes model for which the underlying asset is a **futures** contract
- We will begin by seeing how the Black model can be used to price bond and interest rate options
- Finally, we examine binomial interest rate models, in particular the Black-Derman-Toy model

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- A bond portfolio manager might want to hedge bonds of one duration with bonds of a different duration. This is called **duration hedging**. In general, hedging a bond portfolio based on duration does not result in a perfect hedge
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The Dynamics of Bonds and Interest Rates

- Suppose that the bond-price at time $T - t$ before maturity is denoted by $P(t, T)$ and that it is modeled by the following Ito process:

$$\frac{dP_t}{P_t} = \alpha(r,t)\,dt + q(r,t)\,dZ_t$$

  where

  1. Z is a standard Brownian motion
  2. α and q are coefficients which depend both on time t and the interest rate r

- This aproach requires careful specificatio of the coefficients α and q - and we would like for the model to be simpler ...
- The alternative is to start with the model of the short-term interest rate as an Ito process:

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An Inappropriate Bond-Pricing Model

- We need to be careful when implementing the above strategy.
- For instance, if we assume that the yield-curve is flat, i.e., that at any time the zero-coupon bonds at all maturities have the same yield to maturity, we get that there is possibility for arbitrage
- The construction of the portfolio which creates arbitrage is similar to the one for different Sharpe Ratios and a single source of uncertainty. You should read Section 24.1

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An Equilibrium Equation for Bonds

- When the short-term interest rate is the only source of uncertainty, the following partial differential equation **must be satisfied by any zero-coupon bond** (see equation (24.18) in the textbook)

$$\frac{1}{2}\sigma(r)^2\frac{\partial^2 P}{\partial r^2} + [\alpha(r) - \sigma(r)\phi(r,t)]\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - rP = 0$$

  where

  1. r denotes the short-term interest rate, which follows the Ito process

  $$dr = a(r)dt + \sigma(r)dZ;$$

  2. $\phi(r,t)$ is the Sharpe ratio corresponding to the source of uncertainty Z, i.e.,

  $$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

  with the coefficients $P \cdot \alpha$ and $P \cdot q$ are the drift and the volatility (respectively) of the Ito process P which represents the bond-price for the interest-rate r

- This equation characterizes claims that are a function of the interest rate (as there are no alternative sources of uncertainty).

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The risk-neutral process for the interest rate

- The risk-neutral process for the interest rate is obtained by subtracting the risk premium from the drift:

$$dr_t = [a(r_t) - \sigma(r_t)\phi(r_t, t)]\, dt + \sigma(r_t)\, dZ_t$$

- Given a zero-coupon bond, Cox et al. (1985) show that the solution to the equilibrium equation for the zero-coupon bonds must be of the form (see equation (24.20) in the textbook)

$$P[t, T, r(t)] = \mathbb{E}_t^*[e^{-R(t,T)}]$$

where

1. $\mathbb{E}_t^*$ represents the expectation taken with respect to **risk-neutral** probabilities given that we know the past up to time t;
2. $R(t, T)$ represents the cumulative interest rate over time, i.e., it satisfies the equation (see (24.21) in the book)

$$R(t, T) = \int_t^T r(s)\, ds$$

- Thus, to value a zero-coupon bond, we take the expectation over "all the discount factors" implied by these paths

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Summary

- One approach to modeling bond prices is exactly the same procedure used to price options on stock
- We begin with a model of the interest rate and then use Ito's Lemma to obtain a partial differential equation that describes the bond price - the equilibrium equation
- Next, using the PDE together with boundary conditions, we can determine the price of the bond
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