NAME: 
M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
In-Term Exam II
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 85.
Time: 50 minutes

TRUE/FALSE

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MULTIPLE CHOICE

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Σ
2.1. **CAPM.**

**Problem 2.1.** (10 points) State the assumptions of the Capital Asset Pricing Model.

**Solution:**

I. The market is *competitive*, i.e., the securities are bought and sold at the same price. There are no taxes or transaction costs. Both borrowing and lending are at the risk-free interest rate.

II. Investors hold only efficient portfolios.

III. *Homogeneous expectations:* The investors have the same beliefs about the expected values, volatilities, and correlations of returns of securities.
2.2. TRUE/FALSE QUESTIONS. Please note your answers on the front page.

Problem 2.2. (2 points) Consider our usual coordinate system of portfolios with the volatility on the horizontal axis and the expected return on the vertical axis. Consider a portfolio $P$ in that plane and look at the line through that portfolio and the point corresponding to the risk-free asset $(0, r_f)$. Then, the slope of this line is exactly the Sharpe ratio of the portfolio $P$. True or false?

Solution: TRUE

Problem 2.3. A market maker who delta-hedges completely insures himself against losses. True or false?

Solution: FALSE

Problem 2.4. A market-maker writes a call option on a stock. To decrease the delta of this position, (s)he can write a call on the underlying stock. True or false?

Solution: TRUE

Problem 2.5. (2 points) Market makers usually do not need to rebalance their portfolios after the initial hedge is established. True or false?

Solution: FALSE

Problem 2.6. An efficient portfolio contains only systematic risk. True or false?

Solution: TRUE

Problem 2.7. A portfolio consists of investments indexed by $i = 1, \ldots, n$ whose positive weights are denoted by $x_i, i = 1, \ldots, n$ and whose volatilities are denoted by $\sigma_i, i = 1, \ldots, n$. Let the return of this portfolio be denoted by $R_P$ and let its volatility be denoted by $\sigma_P$. Then,

$$\sigma_P \leq \sum_{i=1}^{n} x_i \sigma_i$$

True or false?

Solution: TRUE

Problem 2.8. Portfolio $P$ has expected return 0.08 and volatility equal to 12%. Portfolio $Q$ has expected return 0.10 and volatility equal to 12.5%. Then, we can say with certainty that portfolio $P$ is not efficient. True or false?

Solution: FALSE

Problem 2.9. The tangent portfolio has the highest Sharpe ratio of all the portfolios in the feasible set. True or false?
Solution: TRUE

Problem 2.10. (2 points) Assume the Black-Scholes stock-price model for the underlying asset of a gap call and an ordinary European call with the strike equal to the trigger of the gap call. The gap call has the trigger price strictly larger than the strike price. Then, the delta of the gap call is always smaller than the delta of the ordinary call. True or false?

Solution: FALSE

The expression for the price of the gap call is

$$v_{GC}(s,t) = se^{-\delta(T-t)} N(d_1(s,t,K_t)) - K_s e^{-r(T-t)} N(d_2(s,t,K_t))$$

with

$$d_1(s,t,K_t) = \frac{1}{\sigma \sqrt{T-t}} \left[ \ln\left( \frac{s}{K_t} \right) + (r - \delta + \frac{1}{2} \sigma^2)(T-t) \right],$$

$$d_2(s,t,K_t) = d_1(s,t,K_t) - \sigma \sqrt{T-t}.$$

with $K_s$ the strike and $K_t$ the payment trigger.

This expression can be rewritten as

$$v_{GC}(s,t) = se^{-\delta(T-t)} N(d_1) - K_s e^{-r(T-t)} N(d_2)$$

$$= se^{-\delta(T-t)} N(d_1(s,t,K_t)) - K_s e^{-r(T-t)} N(d_2(s,t,K_t)) + (K_t - K_s) e^{-r(T-t)} N(d_2(s,t,K_t))$$

$$= v_C(s,t,K_t) + (K_t - K_s) e^{-r(T-t)} N(d_2(s,t,K_t))$$

where $v_C(s,t,K_t)$ denotes the price of the ordinary call with the strike $K_t$.

So, the delta of the gap option is

$$\Delta_{GC}(s,t) = \frac{\partial}{\partial s} v_{GC}(s,t) = \Delta_C(s,t,K_t) + (K_t - K_s) e^{-r(T-t)} \frac{\partial}{\partial s} N(d_2(s,t,K_t))$$

with $\Delta_C(s,t,K_t)$ equal to the delta of an ordinary call with strike $K_t$. Looking at the above expression more closely, you can see that by changing the ordering of $K_t$ and $K_s$ one changes the ordering of the gap’s and the ordinary call’s deltas.
2.3. **FREE-RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they’re correct) are worth 0 points.

**Problem 2.11.** (10 points) Let $S = \{S(t), t \geq 0\}$ denote the price of a non-dividend-paying stock. The stock price today equals $100. Assume that the Black-Scholes setting holds.

Let $r$ denote the continuously compounded risk-free interest rate.

Consider a European call option with exercise date $T = 10$ and strike price $K = S(0)e^{rT}$. You are given that its price today equals $V_C(0) = $68.26.

The goal of this problem is to obtain the implied volatility of the stock $S$.

(i) (5 pts) Write down the expression for the Black-Scholes price of the European call.

(ii) (3 pts) Simplify the expression you obtained in part (i) so that the call price depends only on the volatility $\sigma$.

(iii) (2 pts) Using the properties of the standard normal cumulative distribution function $N$, the standard normal table, the European call price given in the problem and your answer to part (ii), solve for $\sigma$.

**Solution:**

(i) According to the Black-Scholes formula,

$$V_C(0) = V_C(0, S(0), K, r, \delta, \sigma, T) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

with

$$d_1 = \frac{1}{\sigma \sqrt{T}}[\ln(S(0)/K) + (r - \delta + \frac{1}{2}\sigma^2)T],$$

$$d_2 = d_1 - \sigma \sqrt{T}.$$

(ii) Using the provided information, we obtain

$$d_1 = \frac{1}{\sigma \sqrt{T}}[\ln(S(0)/S(0)e^{rT}) + (r + \frac{1}{2}\sigma^2)T]$$

$$= \frac{1}{\sigma \sqrt{T}}[-rT + (r + \frac{1}{2}\sigma^2)T]$$

$$= \frac{1}{2}\sigma \sqrt{T},$$

$$d_2 = -\frac{1}{2}\sigma \sqrt{T} = -d_1.$$

Hence,

$$V_C(\ldots, \sigma) = S(0)N(d_1) - S(0)e^{rT}e^{-rT}N(d_2)$$

$$= S(0)[N(d_1) - N(-d_1)]$$

$$= S(0)[2N(\frac{1}{2}\sigma \sqrt{T}) - 1]$$

$$= 100[2N(\frac{\sqrt{10}}{2}\sigma) - 1].$$
(iii) From the problem, we know that

$$68.26 = 100[2N\left(\frac{\sqrt{10}}{2}\sigma\right) - 1].$$

So,

$$N\left(\frac{\sqrt{10}}{2}\sigma\right) = 0.8413.$$ 

From the standard normal table, we get that

$$\frac{\sqrt{10}}{2}\sigma = 1 \Rightarrow \sigma = \frac{2}{\sqrt{10}} \approx 0.6325.$$
**Problem 2.12.** (10 points) Assume that the risk-free interest rate equals 0.04. The Sharpe ratio of asset $S$ is given to be $1/4$ while the Sharpe ratio of asset $Q$ equals $1/3$. You know that the volatility of $S$ is twice the volatility of $Q$. If you build an equally weighted portfolio with assets $S$ and $Q$ as its two components, the expected return of this portfolio will be 0.10. What is the expected return of $S$ and what is the expected return of $Q$?

**Solution:** From the condition on the Sharpe ratio of $S$, we get

$$\frac{\mathbb{E}[R_S] - r_f}{\sigma_S} = \frac{1}{4} \quad \Rightarrow \quad \sigma_S = 4(\mathbb{E}[R_S] - 0.04).$$

From the condition on the Sharpe ratio of $Q$, we obtain

$$\frac{\mathbb{E}[R_Q] - r_f}{\sigma_Q} = \frac{1}{3} \quad \Rightarrow \quad \sigma_Q = 3(\mathbb{E}[R_Q] - 0.04).$$

Since $\sigma_S = 2\sigma_Q$, we have

$$4(\mathbb{E}[R_S] - 0.04) = 2(3)(\mathbb{E}[R_Q] - 0.04) \quad \Rightarrow \quad 2(\mathbb{E}[R_S] - 0.04) = 3(\mathbb{E}[R_Q] - 0.04)$$

$$\Rightarrow \quad 2\mathbb{E}[R_S] - 3\mathbb{E}[R_Q] = 0.08 - 0.12 = -0.04.$$

On the other hand, from the given expected return on the equally-weighted portfolio, we know that

$$\frac{1}{2}(\mathbb{E}[R_S] + \mathbb{E}[R_Q]) = 0.10 \quad \Rightarrow \quad \mathbb{E}[R_S] + \mathbb{E}[R_Q] = 0.20.$$

Solving the above system of two equations with two unknowns, we get

$$\mathbb{E}[R_S] = 0.112 \quad \text{and} \quad \mathbb{E}[R_Q] = 0.088.$$
Problem 2.13. (10 points) The current price of a non-dividend-paying stock is $25 per share. A market-maker writes a three-month European put option on this stock and proceeds to delta-hedge it. The put premium is $2.50, its delta is $-0.30$, its gamma is $0.04$, and its theta is $-0.01$ per day.

The continuously-compounded, risk-free interest rate is $0.04$.

Assuming that the stock price does not change, what is the approximate overnight profit for the market-maker?

Solution: The initial cost of the total delta-hedged portfolio is

$$-2.50 + (-0.3)(25) = -10.$$ 

The approximate put price after one day is, according to the delta-gamma-theta approximation,

$$2.50 - 0.01 = 2.49.$$ 

So, the overnight profit is

$$-2.49 + (-0.3)(25) + 10e^{0.04/365} = -9.99 + 10e^{0.04/365} = 0.011096.$$
Problem 2.14. (10 points) Your model for the economy at the end of your period has three different states good, so-so and bad. You think that the probability that the economy will be in the so-so state is twice the probability that it will be in the good state. You also think that the probability that the economy will be in the good state is twice the probability that it will be in the bad state.

There are two assets in your market model called $S$ and $Q$. Their returns, depending on the state of the economy are modeled as follows:

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<th>so-so</th>
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<tr>
<td>$S$</td>
<td>10%</td>
<td>2%</td>
<td>-5%</td>
</tr>
<tr>
<td>$Q$</td>
<td>8%</td>
<td>-1%</td>
<td>-4%</td>
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Your portfolio is equally weighted between assets $S$ and $Q$. What is the volatility of this total portfolio?

Solution: The probability of the good economy is $2/7$, the probability of the so-so economy is $4/7$, and the probability of the bad economy is $1/7$. So, the return $R_T$ of the total portfolio has the following distribution

$$R_T = \begin{cases} 
0.09 & \text{with probability } 2/7, \\
0.005 & \text{with probability } 4/7, \\
-0.045 & \text{with probability } 1/7.
\end{cases}$$

We get that the mean return of this portfolio equals

$$\mathbb{E}[R_T] = 0.09 \left( \frac{2}{7} \right) + 0.005 \left( \frac{4}{7} \right) - 0.045 \left( \frac{1}{7} \right) = 0.0221429.$$

The second moment of the portfolio’s return is

$$\mathbb{E}[R_T^2] = (0.09)^2 \left( \frac{2}{7} \right) + (0.005)^2 \left( \frac{4}{7} \right) + (-0.045)^2 \left( \frac{1}{7} \right) = 0.00261786.$$

Therefore, the variance of the return equals

$$Var[R_T] = \mathbb{E}[R_T^2] - (\mathbb{E}[R_T])^2 = 0.00261786 - (0.0221429)^2 = 0.00212755.$$

Finally, the volatility of the portfolio is $\sigma_T = 0.0461254$. 

2.4. MULTIPLE CHOICE QUESTIONS. Please note your answers on the front page.

Problem 2.15. (5 points) Assume the Black-Scholes framework.

The goal is to delta-hedge a written one-year, at-the-money straddle on a non-dividend-paying stock whose current price is $50. The stock’s volatility is 0.20.

The continuously-compounded, risk-free interest rate is 0.10.

What is the cost of delta-hedging the straddle using shares of the underlying stock?

(a) $21.33
(b) $22.58
(c) $24.33
(d) $25.19
(e) None of the above.

Solution: (b)
The $\Delta$ of the straddle equals

$$2\Delta_C - 1 = 2N(d_1) - 1$$

with

$$d_1 = \frac{1}{0.2}(0.10 + 0.02) = 0.6.$$ 

Our answer is

$$50(2N(0.60) - 1) = 50(2(0.7257) - 1) = 50(0.4515) = 22.575.$$ 

Problem 2.16. (5 points) Consider a two-year project, where the cost of capital is 4%. There are only three cash flows for this project:

- The first occurs at $t = 0$, and is $-100$.
- The second occurs at $t = 1$, and is $-50$.
- The third occurs at $t = 2$, and is $X$.

Determine $X$, the level of the cash flow at $t = 2$, that leads to the project breaking even.

(a) -$160.16
(b) -$56.16
(c) $56.16
(d) $160.16
(e) None of the above.

Solution: (d)

$$X = 100(1.04)^2 + 50(1.04) = 160.16.$$
Problem 2.17. (5 points) Consider two assets $X$ and $Y$ such that:
- their expected returns are $\mathbb{E}[R_x] = 0.12$ and $\mathbb{E}[R_y] = 0.15$;
- their volatilities are $\sigma_x = 0.3$ and $\sigma_y = 0.2$;
- the correlation coefficient of their returns is $\rho_{X,Y} = -1$.
You are tasked with constructing a portfolio consisting of shares of $X$ and $Y$ with a risk-free return. What is the risk-free return $r_f$?
(a) 0.132
(b) 0.135
(c) 0.138
(d) Not enough information is given.
(e) None of the above.

Solution: (c)
In the portfolio consisting of $X$ and $Y$ with the risk-free expected return, the weight of asset $X$ equals
$$ w_X = \frac{\sigma_Y}{\sigma_X + \sigma_Y} = \frac{0.2}{0.3 + 0.2} = 0.4. $$
So, the total portfolio’s return is
$$ r_f = 0.4(0.12) + 0.6(0.15) = 0.138. $$

Problem 2.18. A market-maker sells option I for $10. This option’s delta is 0.6557 and its gamma is 0.02. The market maker proceeds to delta-gamma hedge this commitment by trading in the underlying and also in option II on the same stock. The latter option’s price is $4.70, its delta is 0.5794 and its gamma is 0.04.
What is the market-maker’s resulting position in option II?
(a) Buy 0.5 of option II.
(b) Write 0.5 of option II.
(c) Buy 2 of option II.
(d) Write 2 of option II.
(e) None of the above.

Solution: (a)
With $n_{II}$ denoting the position in option II, to achieve gamma-neutrality we need
$$ -0.02 + n_{II}(0.04) = 0 \quad \Rightarrow \quad n_{II} = 0.5. $$