UNIVERSITY OF TEXAS AT AUSTIN

Lecture 2
Black-Derman-Toy.

2.1. **The Black-Derman-Toy (BDT) Tree.** The basic idea of the BDT model is to compute a binomial tree of short-term interest rates, with a flexible enough structure to match the data. Consider market information about bonds that we would like to match; namely, we would like to match (effective annual) yield to maturity, bond prices and the volatility of the bond **yields** (see Table 24.2 in the textbook). Black, Derman, and Toy describe their tree as driven by the short-term rate, which they assume is **lognormally distributed**. For each period in the tree, there are two parameters:

- R_{ih} is a **rate level parameter** at a given time; and
- σ_i is a **volatility** parameter

where h stands for the length of every period while i denotes the number of periods that have elapsed so far in the tree. These parameters are chosen to match the tree with the observed data. The general form of the BDT tree is displayed in Figure 24.4 in the textbook.

Remark 2.1. The models we have examined are arbitrage-free in a world consistent with their assumptions. This world is our *market-model*. In the real world, however, we recognize the following phenomenon. Every parametric model worth considering is going to be relatively sparse in the sense that it has a relatively small number of parameters. These parameters are frequently estimated using observations. This process of matching a model to fit the data is called **calibration**. By design, we cannot use all of the observed prices in calibration. The model is a simplified reflection of the whole world and it cannot include all of the prices present in reality. In particular, the Black-Derman-Toy (BDT) tree is a binomial interest rate tree calibrated to match zero-coupon bond yields and a particular set of volatilities. We will notice that the model generates **apparent** arbitrage opportunities, i.e., observed prices that do not match theoretical prices obtained using the model. This is not suprising and is a characteristic of just about any predictive statistical model.

2.1.1. *The construction.* Let the time-h price of a zero-coupon bond maturing at time T if the time-h short-term rate is $r(h)$ be denoted by $P[h, T, r(h)]$. Then, the **annualized** yield of this bond is

$$y[h, T, r(h)] = P[h, T, r(h)]^{-1/(T-h)} - 1$$

Assume that at time h, the short-term rate $r(h)$ can take on the two values: r_u and r_d. Then, the annualized lognormal yield volatility (the variance of a linear transformation of a Bernoulli random variable, really!) equals

$$\frac{1}{2\sqrt{h}} \ln\left(\frac{y(h, T, r_u)}{y(h, T, r_d)}\right)$$

Here is the way to populate the BDT tree moving from the left to the right so that it is consistent with:

(1) the **observed** terms structure,
(2) our **specification** (maybe an estimate) of the volatility.

It is customary to assume that the risk-neutral probability in a BDT tree equals $p = 1/2$. We start at early nodes and work to the later nodes, building the tree "outward", i.e., from the parent nodes to the offspring nodes and one time-period at a time.

The first node is given by the prevailing 1-year spot rate we temporarily denote by R_0. If the R_0 itself is provided, we write it in the root node of the interest-rate tree. Let the observed one-dollar, 1-year zero-coupon bond price be denoted by P_0. The spot rate and the bond price need to be consistent with one another in the following way

$$P_0 = 1/(1 + R_0).$$

Whence, if the bond prices are provided, we solve for R_0 above and write it in the tree.

For the up/down nodes at the end of the first period, let the year-1 price of a 1-year bond be denoted by $P(1, 2, r_u)$ or $P(1, 2, r_d)$, depending on the movement of the interest rate. Let P_1 be the **observed** year-1 price of a 1-year bond. If σ_1 denotes the volatility of the interest rates at time-1, then in our model, we can follow this notational convention $r_u = r_d e^{2\sigma_1}$. We require these two consistency conditions be satisfied:

$$\begin{aligned} P_1 &= \frac{1}{1+R_0}\left[\frac{1}{2}P(1,2,r_u) + \frac{1}{2}P(1,2,r_d)\right] = \frac{1}{1+R_0}\left[\frac{1}{2(1+r_u)} + \frac{1}{2(1+r_d)}\right] \\ &= \frac{1}{1+R_0}\left[\frac{1}{2(1+r_d e^{2\sigma_1})} + \frac{1}{2(1+r_d)}\right], \\ \sigma_1 &= \frac{1}{2}\ln\left(\frac{P(1,2,r_u)^{-1}-1}{P(1,2,r_d)^{-1}-1}\right) = \frac{1}{2}\ln\left(\frac{r_u}{r_d}\right) = \frac{1}{2}\ln\left(\frac{r_d e^{2\sigma_1}}{r_d}\right). \end{aligned}$$

If the zero-coupon bond prices at time-0 are given, the above is a system of two equations with two unknowns r_d and σ_1. We solve for the two and continuw to follow this recipe to enter values at the remaining nodes for each subsequent period.

Example 2.2. BDT tree from specified volatilities
Let the current effective annual spot rates be

$$r_0(0,1) = 0.04, \quad r_0(0,2) = 0.045, \quad r_0(0,3) = 0.05.$$

Additionally, we assume that the BDT tree is constructed under the assumption that the volatility of the annual effective one-year spot rates in one year is $\sigma_1 = 0.08$ and that the volatility of the annual effective one-year spot rates in two years is $\sigma_2 = 0.10$.

At the end of the first period, we find the values of r_u and r_d consistent with the above specified spot rates and volatility $\sigma_1 = 0.08$. From the two-year spot rate, we have

$$\frac{1}{(1+r_0(0,2))^2} = \frac{1}{1+r_0(0,1)} \times \frac{1}{2}\left[\frac{1}{1+r_d e^{2\sigma_1}} + \frac{1}{1+r_d}\right].$$

So,

$$\frac{1}{(1.045)^2} = \frac{1}{1.04} \times \frac{1}{2}\left[\frac{1}{1+r_d e^{0.16}} + \frac{1}{1+r_d}\right] \quad \Rightarrow \quad \frac{2.08}{1.045^2} = \frac{1}{1+r_d e^{0.16}} + \frac{1}{1+r_d}.$$

We solve for r_d in the following equation:

$$1.9(1+r_d)(1+r_d e^{0.16}) = 1 + r_d e^{0.16} + 1 + r_d = 2 + r_d(1+e^{0.16}).$$

The quadratic in r_d becomes

$$2.23r_d^2 + 1.96r_d - 0.1 = 0 \quad \Rightarrow \quad r_d = 0.0483.$$

So, the interest rate at the up node equals $r_u = r_d e^{2\sigma_1} = 0.0483e^{0.16} = 0.0568$.

At the end of the second period, we need to figure out the interest rates r_{uu}, r_{ud} and r_{dd} (the BDT is recombining). According to the BDT model, if the volatility σ_2 of interest rates at time-2 is given, we can write $r_{ud} = r_{dd}e^{2\sigma_2}$ and $r_{ud} = r_{dd}e^{4\sigma_2}$.

Using the given three-year spot rate, we get

$$\frac{1}{(1+r_0(0,3))^3} = \frac{1}{(1+r_0(0,2))^2} \times \frac{1}{2} \times \left[\frac{1}{2} \times \frac{1}{1+r_{dd}e^{4\sigma_2}} + \frac{1}{1+r_{dd}e^{2\sigma_2}} + \frac{1}{2} \times \frac{1}{1+r_{dd}}\right].$$

So,

$$\frac{2 \times 1.045^2}{(1.05)^3} = \frac{1}{2} \times \frac{1}{1+r_{dd}e^{0.4}} + \frac{1}{1+r_{dd}e^{0.2}} + \frac{1}{2} \times \frac{1}{1+r_{dd}}.$$

The above is a job for software. So, we enlist *Mathematica*.

$$\text{NSolve}\left[\frac{2 * 1.045^2}{1.05^3} = \left(\frac{0.5}{e^{0.4}x+1} + \frac{1}{e^{0.2}x+1}\right) + \frac{0.5}{x+1}, x\right]$$

The answers we obtain are

$$\{\{x \to -0.947202\}, \{x \to -0.713977\}, \{x \to 0.0487493\}\}.$$

We discard the first two answers as they are not acceptable interest rates. Therefore, $r_{dd} = 0.0487, r_{du} = 0.0595$, and $r_{uu} = 0.0727$.

Example 2.3. A call on a bond

Using the above BDT tree, let us find the price of a two-year, at-the-money call option on a three-year, zero-coupon bond redeemable for \$1.

First, we figure out the strike of this call option. Today's price of our bond can be calculated using the spot price $r_0(0,3)$. We get

$$(1+r_0(0,3))^{-3} = (1.05)^{-3} = 0.8638.$$

In three years, the bond will be a one-year bond with one of the following three values, depending on the interest rate at a particular node in the tree:

$$\begin{aligned} P_{uu} &= (1+r_{uu})^{-1} = (1.0727)^{-1} = 0.9322, \\ P_{ud} &= (1+r_{ud})^{-1} = (1.0727)^{-1} = 0.9438, \\ P_{du} &= (1+r_{du})^{-1} = (1.0727)^{-1} = 0.9438, \\ P_{dd} &= (1+r_{dd})^{-1} = (1.0727)^{-1} = 0.9536. \end{aligned}$$

So, the call's payoffs are

$$V_{uu} = 0.0684, \quad V_{ud} = 0.08, \quad V_{du} = 0.08, \quad V_{dd} = 0.0898.$$

Finally, the call's time-0 price is

$$\begin{aligned} V_C(0) &= (1+r_0(0,2))^{-2} \times \frac{1}{2} \times \left[\frac{1}{2} \times V_{uu} + V_{ud} + \frac{1}{2} \times V_{dd}\right] \\ &= 1.045^{-2} \times 0.5 \times [0.5 \times 0.0684 + 0.08 + 0.5 \times 0.0898] = 0.0728. \end{aligned}$$

2.1.2. *Verification.* First, you enter the market data into the BDT tree - using the expressions at the nodes of the generic BDT tree and plugging in the data - which depicts the 1-year effective annual rates (see Figure 24.5). The tree will be different from the binomial trees we have seen so far, e.g., Unlike a stock-price tree, the nodes are not necessarily "centered" on the previous periods nodes; this is because the tree is matching the **data** by construction. One can verify that the recipe for the values at the nodes given in the generic BDT tree is indeed consistent with the data:

1. To verify that the tree matches the yield curve, one should compute the prices of zero-coupon bonds with maturities of 1, 2, 3, and 4 years.
2. To verify the volatilities, one should compute the prices of $1-$, $2-$, and 3-year zero-coupon bonds at year 1, and then compute the yield volatilities of those bonds.

You can read more on the verification steps in the textbook as a reading assignment.