Scaled Random Walks

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2. Scaled Symmetric Random Walk
3. Log-Normal Distribution as the Limit of the Binomial Model

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Construction

- The goal is to create a Brownian motion
- We begin with a symmetric random walk, i.e., we repeatedly toss a fair coin ($p = q = 1/2$)
- Let X_j be the random variable representing the outcome of the j^{th} coin toss in the following way

$$X_j = \begin{cases} 1 & \text{if the outcome is heads} \\ -1 & \text{if the outcome is tails} \end{cases} \quad \text{for } j = 1, 2, \ldots$$

- Define $M_0 = 0$ and

$$M_k = \sum_{j=1}^{k} X_j, \text{ for } k = 1, 2, \ldots$$

- We call the process $M_k, k = 0, 1, \ldots$ a symmetric random walk

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Increments of the symmetric random walk

- A random walk has **independent increments**, i.e., for every choice of nonnegative integers $0 = k_0 < k_1 < \cdots < k_m$, the random variables

$$M_{k_1} - M_{k_0},\ M_{k_2} - M_{k_1},\ \ldots\ M_{k_m} - M_{k_{m-1}}$$

  are **independent**
- Each of the random variables

$$M_{k_{i+1}} - M_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j$$

  is called an increment of the random walk
- The **expected value** of each increment is 0
- As for the **variance**, we have

$$Var[M_{k_1+1} - M_{k_i}] = \sum_{j=k_i+1}^{k_{i+1}} Var[X_j] = \sum_{j=k_i+1}^{k_{i+1}} 1 = k_{i+1} - k_i$$

- We say that the variance of the symmetric random walk accumulates at the rate one per unit time

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Quadratic Variation of the Symmetric Random Walk

- Consider the quadratic variation of the symmetric random walk, i.e.,

$$[M, M]_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2 = k.$$

- Note that the quadratic variation is computed **path-by-path**
- Also note that **seemingly** the quadratic variation $[M, M]_k$ equals the variance of M_k - but these are computed in a different fashion and have different meanings:
- The variance is an average over all possible paths; it is a **theoretical** quantity
- The quadratic variation is evaluated with a **single path** in mind; from tick-by-tick price data, one can calculate the quadratic variation for any **realized** path

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The definition

- Recall the illustrative graphs on the convergence of random walks …
- For a **fixed** integer n, we define the scaled symmetric random walk by

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$$

for all $t \geq 0$ such that nt is an integer; for all other nonnegative t - we define $W^{(n)}(t)$ by **linear interpolation**

- The scaled random walk has **independent increments**, i.e., if $0 = t_0 < t_1 < \cdots < t_m$ are such that nt_j is an integer for all j, then the random variables

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More properties of the scaled symmetric random walk

- Let $0 \leq s \leq t$ be such that both ns and nt are integers, then

$$\mathbb{E}[W^{(n)}(t) - W^{(n)}(s)] = 0$$

$$Var[W^{(n)}(t) - W^{(n)}(s)] = t - s$$

- The quadratic variation for any t such that nt is an integer equals

$$[W^{(n)}, W^{(n)}](t) = \sum_{j=1}^{nt} \left[W^{(n)}(\frac{j}{n}) - W^{(n)}(\frac{j-1}{n}) \right]^2$$

$$= \sum_{j=1}^{nt} \left[\frac{1}{\sqrt{n}} X_j \right]^2$$

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Limiting Distribution of the Scaled Random Walk

- Central Limit Theorem

  Fix $t \geq 0$. As $n \to \infty$, the distribution of the scaled random walk $W^{(n)}(t)$ evaluated at time t converges to the normal distribution with mean zero and variance t, i.e., for every $t \geq 0$

$$W^{(n)}(t) \Rightarrow N(0, t)$$

- We use the CLT in statements such as:

$$\mathbb{E}[g(W^{(100)}(0.25))] \approx \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-2x^2}\, dx$$

  where g is any continuous, bounded function

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Scaled Random Walks

Back to the binomial pricing model

- The Central Limit Theorem allows us to conclude that the limit of a properly scaled binomial asset-pricing model leads to a stock-price with a log-normal distribution.
- Consider a binomial model for a stock price on the time interval $[0, t]$ with n steps (binomial periods); assume that n and t are chosen so that nt is an integer
- Let the "up factor" be $u_n = 1 + \frac{\sigma}{\sqrt{n}}$ and let the "down factor" be $d_n = 1 - \frac{\sigma}{\sqrt{n}}$
- For simplicity, assume that there is no interest rate and that the stock pays no dividends. The final result will hold in those cases as well, but it is a bit harder to exhibit
- The risk neutral probabilities are

$$p^* = \frac{1 - d_n}{u_n - d_n} = \frac{1}{2}, \; q^* = 1 - p^* = \frac{1}{2}$$

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Coin tosses

- $S(0)\ldots$ initial stock price
- $H_{nt}\ldots$ the number of heads in the first nt coin tosses
- $T_{nt}\ldots$ the number of tails in the first nt coin tosses, i.e.,

$$T_{nt} = nt - H_{nt}$$

- Then, the **symmetric random walk** M_{nt} is the number of heads minus the number of tails, i.e.,

$$M_{nt} = H_{nt} - T_{nt}$$

- Hence,

$$H_{nt} = \frac{1}{2}(nt + M_{nt}) \text{ and } T_{nt} = \frac{1}{2}(nt - M_{nt})$$

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Binomial stock-price. Convergence Theorem

- Using the above notation and the binomial pricing model, we get that the stock price at time t equals

$$\begin{aligned} S_n(t) &= S(0) u_n^{H_{nt}} d_n^{T_{nt}} \\ &= S(0) \left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt + M_{nt})} \left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt - M_{nt})} \end{aligned}$$

- **Theorem.**
  As $n \to \infty$, the distribution of $S_n(t)$ converges to the distribution of

$$S(t) = S(0) \exp\left\{\sigma W(t) - \frac{1}{2}\sigma^2 t\right\}$$

  where $W(t)$ is a normal random variable with mean zero and variance t.

- The above is very important !!!!!

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The log-normal distribution

- **Definition.** The distribution of $S(t)$ is called log-normal.
- In general, any random variable of the form ce^X with c a constant and X a normally distributed random variable is referred to as log-normal
- In the present case,

$$X = \sigma W(t) - \frac{1}{2}\sigma^2 t$$

  is normal with mean $-\frac{1}{2}\sigma^2 t$ and variance $\sigma^2 t$

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- In the present case,

$$X = \sigma W(t) - \frac{1}{2}\sigma^2 t$$

is normal with mean $-\frac{1}{2}\sigma^2 t$ and variance $\sigma^2 t$

The log-normal distribution

- **Definition.** The distribution of $S(t)$ is called log-normal.
- In general, any random variable of the form ce^X with c a constant and X a normally distributed random variable is referred to as log-normal
- In the present case,

$$X = \sigma W(t) - \frac{1}{2}\sigma^2 t$$

is normal with mean $-\frac{1}{2}\sigma^2 t$ and variance $\sigma^2 t$