Understanding Risk-Neutral Probability

Risk-Neutral Pricing Revisited

- We can interpret the terms $(e^{(r-\delta)h} - d)/(u - d)$ and $(u - e^{(r-\delta)h})/(u - d)$ as probabilities
- Let

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} \tag{10.5}$$

- Then equation (10.3) can then be written as

$$C = e^{-rh}[p^* C_u + (1 - p^*)C_d] \tag{10.6}$$

  - Where p^* is the **risk-neutral probability** of an increase in the stock price

Understanding Risk-Neutral Pricing

- A **risk-neutral** investor is indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing (compare to risk-averse)
- p^* is the risk-neutral probability that the stock price will go up

Understanding Risk-Neutral Pricing (Cont'd)

- The option pricing formula can be said to price options as *if* investors are risk-neutral
  - Note that we are not assuming that investors are actually risk-neutral, and that risky assets are actually expected to earn the risk-free rate of return

Pricing an Option Using Real Probabilities

- Is option pricing consistent with standard discounted cash flow calculations?

  Yes.

  However, discounted cash flow is not used in practice to price options

Pricing an Option Using Real Probabilities (cont'd)

- Suppose that the continuously compounded expected return on the stock is α and that the stock does not pay dividends
- If p is the true probability of the stock going up, p must be consistent with u, d, and α

$$puS + (1 - p)dS = e^{\alpha h}S \tag{11.3}$$

- Solving for p gives us

$$p = \frac{e^{\alpha h} - d}{u - d} \tag{11.4}$$

Pricing an Option Using Real Probabilities (cont'd)

- Using *p*, the actual expected payoff to the option one period hence is

$$pC_u + (1-p)C_d = \frac{e^{\alpha h} - d}{u - d} C_u + \frac{u - e^{\alpha h}}{u - d} C_d \qquad (11.5)$$

- At what rate do we discount this expected payoff?
  - It is not correct to discount the option at the expected return on the stock, α, because the option is equivalent to a leveraged investment in the stock and hence is riskier than the stock

Pricing an Option Using Real Probabilities (cont'd)

- Denote the appropriate per-period discount rate for the option as γ
- Since an option is equivalent to holding a portfolio consisting of Δ shares of stock and B bonds, the expected return on this portfolio is

$$e^{\gamma h} = \frac{S\Delta}{S\Delta + B} e^{\alpha h} + \frac{B}{S\Delta + B} e^{rh} \tag{11.6}$$

Pricing an Option Using Real Probabilities (cont'd)

- We can now compute the option price as the expected option payoff, discounted at the appropriate discount rate, given by equation (11.6). This gives

$$C = e^{-\gamma h}\left[\frac{e^{\alpha h} - d}{u - d} C_u + \frac{u - e^{\alpha h}}{u - d} C_d\right] \tag{11.7}$$

Pricing an Option Using Real Probabilities (cont'd)

- It turns out that this gives us the same option price as performing the risk-neutral calculation
  - Note that it does not matter whether we have the "correct" value of α to start with
  - Any consistent pair of α and γ will give the same option price
  - Risk-neutral pricing is valuable because setting $\alpha = r$ results in the simplest pricing procedure.