Notes: This is a closed book and closed notes exam.

Time: 75 minutes

**TRUE/FALSE**

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2)</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
<tr>
<td>2</td>
<td>(2)</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
<tr>
<td>3</td>
<td>(2)</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
<tr>
<td>4</td>
<td>(2)</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
<tr>
<td>5</td>
<td>(2)</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
<tr>
<td>6</td>
<td>(2)</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
<tr>
<td>7</td>
<td>(2)</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
<tr>
<td>8</td>
<td>(2)</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
<tr>
<td>9</td>
<td>(2)</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
</tbody>
</table>

**MULTIPLE CHOICE**

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(5)</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>2</td>
<td>(5)</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>3</td>
<td>(5)</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>4</td>
<td>(5)</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>5</td>
<td>(5)</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>6</td>
<td>(5)</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
</tbody>
</table>

**FOR GRADER’S USE ONLY:**

<table>
<thead>
<tr>
<th>T/F</th>
<th>1.</th>
<th>2.</th>
<th>M.C.</th>
<th>Σ</th>
</tr>
</thead>
</table>
Part I. **TRUE/FALSE QUESTIONS**

*Please note your answers on the front page.*

1. You are trying to estimate the mean rate of return on a stock. Then, using more frequent observations of the stock price produces a more accurate estimate.
   **Solution:** FALSE

2. The symmetric simple random walk accumulates quadratic variation at rate one per unit of time.
   **Solution:** TRUE

3. Continuously compounded returns of stocks are multiplicative.
   **Solution:** FALSE
   See p. 353 in McDonald.

4. The product of any pair of lognormal random variables is also lognormally distributed.
   **Solution:** FALSE
   Let $X$ be a lognormal random variable. Let us write $X = e^\eta$ where $\eta$ is a normal random variable. Then, $Y = 1/X$ is also lognormal since it can be written as $Y = e^{-\eta}$. However, $XY = 1$ — not a lognormal random variable.

5. A simple symmetric random walk has independent increments.
   **Solution:** TRUE

6. A simple symmetric random walk is an appropriate model for the evolution of stock prices.
   **Solution:** FALSE
   The simple symmetric random walk can have negative values, while the stock prices cannot become negative.

7. In our usual notation, the time—t forward price of a bond deliverable at $T$ is
   $$F_{t,T}[P(T, T + s)] = \frac{P(t, T + s)}{P(t, T)}.$$
   **Solution:** TRUE

8. In our usual notation, let $S(0) = 110, S(1/2) = 118$ and let $S(1) = 114$ be the observed prices of a share of stock. Then, the estimate of the mean continuously compounded annual rate of return of this stock $\hat{\alpha}$ is less than 3%.
   **Solution:** FALSE
   $$\hat{\alpha} = \ln (114/110) \approx 0.0357.$$

9. A caplet is a financial instrument used as protection against the increase in the interest rate for all repayment installments of a loan to be repaid over multiple periods.
   **Solution:** FALSE
   The caplet only applies to one installment; it’s the cap that provides protection over multiple periods.
10. Assume the Black-Scholes stock-pricing model is in force. Let $\mathbb{E}^*$ denote the expectation under the risk-neutral probability measure $\mathbb{P}^*$. Let $\{S(t), t \geq 0\}$ denote the price of a continuous-dividend-paying stock. Then, in our usual notation,$$
abla^* S(T) = S(0)e^{(r-\delta)T}.$$

**Solution:** TRUE

11. In the Black-Derman-Toy model, the interest rate at any node is the geometric average of the rates at the two nodes at adjacent heights.

**Solution:** TRUE

12. The Black-Scholes option pricing formula can **always** be used for pricing American-type call options on non-dividend-paying assets.

**Solution:** TRUE

---

**Part II. Free-response problems**

Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they’re correct) are worth 0 points.

1. (25 points) Let $S(0) = $120, $K = $100, $\sigma = 0.3$, $r = 0$ and $\delta = 0.08$.

   a. (10 pts) Let $V_C(0, T)$ denote the Black-Scholes European call price for the maturity $T$. Does the limit of $V_C(0, T)$ as $T \to \infty$ exist? If it does, what is it?

   b. (10 pts) Now, set $r = 0.001$ and let $V_C(0, T, r)$ denote the Black-Scholes European call price for the maturity $T$. Again, how does $V_C(0, T, r)$ behave as $T \to \infty$?

   c. (5 pts) Interpret in a sentence or two the differences, if any, between your answers to questions in a. and b.

**Solution:**

a. By the Black-Scholes pricing formula, the function $V_C(0, T)$ has the form

$$V_C(0, T) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) = S(0)e^{-\delta T}N(d_1) - KN(d_2),$$

where $N$ denotes the distribution function of the unit normal distribution and

$$d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + \left( -\delta + \frac{1}{2} \sigma^2 \right) T \right],$$

$$d_2 = d_1 - \sigma \sqrt{T}.$$

As $T \to \infty$, we have that

$$e^{-\delta T} N(d_1) \leq e^{-\delta T} \to 0,$$

$$d_2 \to -\infty \Rightarrow N(d_2) \to 0.$$

Hence,

$$V_C(0, T) \to 0, \text{ as } T \to \infty.$$
b. In this case, the price of the call option reads as

\[ V_C(0, T, r) = S e^{-\delta T} N(d_1) - K e^{-r T} N(d_2), \]

with

\[
d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + \left( r - \delta + \frac{1}{2} \sigma^2 \right) T \right],
\]
\[
d_2 = d_1 - \sigma \sqrt{T}.
\]

As \( T \to \infty \), we have that

\[
e^{-\delta T} N(d_1) \leq e^{-\delta T} \to 0,
\]
\[
d_2 \to -\infty \Rightarrow N(d_2) \to 0.
\]

Since the function \( N \) is bounded between 0 and 1, we see that as \( T \to \infty \), \( V_C(0, T, r) \to 0 \).

c. Until the call option is exercised, the owner of the option can earn interest on the strike price which he/she can invest at the risk-free rate. However, in forfeiting the physical ownership of the asset, he/she also forfeits the possible dividend payments. In part b., the interest rate was reintroduced, but it was still much smaller than the dividend yield. In both cases, the longer the life of the option, the more dividend payments are forfeited and the value of the option itself becomes negligible. It would be interesting to look at the above problem if we assume \( \delta \ll r \) and let \( T \to \infty \).
2. Consider the following Black-Derman-Toy interest-rate tree modeling the future evolution of annual **effective** interest rates. The period-length is one year.

(i) (5 points) Calculate the interest rate denoted in the tree by $r_{dd}$.

(ii) (5 points) Assume that the volatility of the effective interest rates for the second year equals $\sigma_1 = 0.1$. Calculate $r_d$ in the above tree.

(iii) (10 points) Consider a two-year interest rate cap with the notional amount of $1,000 and an annual effective cap rate of $K_R = 0.05$. Calculate the price of this interest rate cap.

**Solution:**

(i) According to the Black-Derman-Toy model, we have

$$\frac{0.06}{0.05} = \frac{0.05}{r_{dd}} \Rightarrow r_{dd} = \frac{0.05^2}{0.06} = 0.0417.$$

(ii) Again, in the BDT tree, we have

$$r_u = r_d e^{2\sigma_1} \Rightarrow r_d = 0.055 e^{-2 \cdot 0.1} = 0.045.$$
(iii) This is a two year cap, so there is only one node at which payment can take place: the “up”-node. The price is
\[
\frac{1}{2} \times \frac{1}{1.05} \times \frac{1}{1.055} \times (0.055 - 0.05) \times 1000 = 2.26.
\]

Part III. MULTIPLE CHOICE QUESTIONS

Please note your answers on the front page.

1. Consider the following binomial interest-rate tree modeling the future evolution of annual continuously compounded interest rates. The period-length is one year.

The risk-neutral probability is given to be equal to 1/2. What is the price of a zero-coupon bond redeemable in three years for $1,000?

(a) $814.85.
(b) $860.72.
(c) $898.78.
(d) $904.85.
(e) None of the above.
Solution: (b)
The price of a one-dollar, zero-coupon bond is

\[ P_0(0, 3) = e^{-0.05} \times \frac{1}{4} \times (e^{-0.045}(e^{-0.04} + e^{-0.05}) + e^{-0.055}(e^{-0.05} + e^{-0.06})) = 0.860719. \]

So, the answer is $860.72.

2. The current stock price is $90. According to your coworker, the time−1 price of this stock will be uniformly distributed between $80 and $120. What is the expected rate of return on this stock for this one year period and the model she is suggesting?

(a) About 12%.
(b) About 11.5%.
(c) About 11%.
(d) About 10.5%.
(e) None of the above.

Solution: (d)
The expected rate of return \( \alpha \) satisfies

\[ S(0)e^{\alpha} = E[S(1)] \Rightarrow \alpha = \ln\left(\frac{100}{90}\right) = \ln(10/9) = 0.1053. \]

3. For a stock price that was initially $55.00, what is the price after 4 years if the continuously compounded returns for these 4 years are 4.5%, 6.2%, 8.9%, and −3.2%?

(a) $59.08
(b) $64.80
(c) $74.80
(d) $84.10
(e) None of the above.

Solution: (b)

\[ 55e^{0.045+0.062+0.089-0.032} \approx 64.80. \]

4. The current price of a continuous-dividend paying stock is observed to be $50 per share while its volatility is given to be 0.34. The dividend yield is projected to be 0.02. The continuously compounded, risk-free interest rate is 0.05.

Consider a European call option with the strike price equal to $40 and the exercise date in three months.

Using the Black-Scholes pricing formula, find the value \( V_{C}(0) \) of this option at time−0.

(a) $9.08
(b) $9.80
(c) $10.55
(d) $14.10
(e) None of the above.
Solution: (c)
In our usual notation,
\[ d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + \left( r - \delta + \frac{1}{2} \sigma^2 \right) T \right] \]
\[ = \frac{1}{0.34 \sqrt{\frac{1}{4}}} \left[ \ln \left( \frac{50}{40} \right) + \left( 0.05 - 0.02 + \frac{1}{2} \times 0.34^2 \right) \times \frac{1}{4} \right] = 1.44, \]
\[ d_2 = d_1 - \sigma \sqrt{T} = 1.27. \]
The standard normal tables give us
\[ N(d_1) = 0.9253, \quad N(d_2) = 0.8983. \]
Finally,
\[ V_C(0) = S(0)e^{-rT}N(d_1) - Ke^{-rT}N(d_2) = 10.55. \]

5. In this problem, use the Black-Scholes pricing model.
Consider a bear spread consisting of a 20−strike put and a 25−strike put. Suppose that \( \sigma = 0.30, r = 0.04, \delta = 0, \) \( T = 1 \) and \( S(0) = 15. \)
What is the price of this bear spread?
(a) About $4.36
(b) About $4.80
(c) About $9.16
(d) About $13.96
(e) None of the above

Solution: (a)
Here, you sell a 20−strike put and buy a 25−strike put.
For the 20−strike put, we have
\[ d_1 = \frac{1}{0.3 \cdot 1} \left[ \ln \left( \frac{15}{20} \right) + (0.04 + \frac{1}{2} \cdot 0.3^2) \cdot 1 \right] = -0.6756; \]
\[ d_2 = -0.9756. \]
The price of the 20−strike put is
\[ V_P(0, 20) = Ke^{-rT}N(-d_2) - S(0)e^{-\delta T}N(-d_1) = 4.79698. \]
For the 25−strike put, we have
\[ d_1 = \frac{1}{0.3 \cdot 1} \left[ \ln \left( \frac{15}{25} \right) + (0.04 + \frac{1}{2} \cdot 0.3^2) \cdot 1 \right] = -1.4194; \]
\[ d_2 = -1.7194. \]
The price of the 25−strike put is
\[ V_P(0, 25) \approx 25e^{-0.04}N(-d_2) - 15e^{-0.02}N(-d_1) = 9.16076 \]
So, the price of the bear spread is
\[ V_P(0, 25) - V_P(0, 20) = 9.16076 - 4.79698 = 4.36378. \]
6. Assume the Black-Scholes framework. Let the current price of a share of non-dividend-paying stock be equal to \( S(0) = 100 \); let its volatility be \( \sigma = 0.3 \).

Consider a gap call option with expiration date \( T = 1 \), with the trigger price 100 and the strike price 90.

You are given that the continuously compounded risk-free interest rate equals \( r = 0.04 \) per annum.

Let \( V_{GC}(0) \) denote the price of the above gap option. Then,
(a) \( V_{GC}(0) < $13.20 \)
(b) \( $13.20 \leq V_{GC}(0) < $15.69 \)
(c) \( $15.69 \leq V_{GC}(0) < $17.04 \)
(d) \( 17.04 \leq V_{GC}(0) < $17.04 \)
(e) None of the above.

**Solution:** (d)

In our usual notation, the Black-Scholes formula for the price of a gap call option reads as

\[
V_{GC}(0) = S(0)^{-\delta T}N(d_1) - K_1 e^{-rT}N(d_2)
\]

where

\[
d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln(S(0)/K_2) + (r - \delta + \frac{1}{2} \sigma^2)T \right],
\]

\[
d_2 = d_1 - \sigma \sqrt{T}.
\]

In the present problem,

\[
d_1 = \frac{1}{0.3} \left[ \ln(100/100) + (0.04 + \frac{0.09}{2}) \right] \approx 0.28;
\]

\[
d_2 = -0.02.
\]

So, the price equals

\[
V_{GC}(0) = 100N(0.28) - 90e^{-0.04}N(-0.02) = 18.49.
\]

7. Assume the Black-Scholes setting. Alice wagers to pay one share of stock to Bob if the price of non-dividend-paying stock in 1 year is above $100.00. Assume \( S(0) = 100.00, \sigma = 0.3, \) and \( r = 0.04 \). What is the time–0 value of this bet?
(a) About $61.15
(b) About $81.15
(c) About $91.15
(d) About $100
(e) None of the above.

**Solution:** (a) or (e)

The Black-Scholes price of this asset call is

\[
V^3[S(0), 0] = S(0)N(d_1)
\]
with 
\[ d_1 = \frac{1}{0.3} \left( \ln \left( \frac{100}{100} \right) + 0.04 + \frac{1}{2} \cdot 0.09 \right) = 0.2833. \]

So,
\[ V^3[S(0), 0] = 100N(0.28) = 61.03. \]

*Note:* If you use the standard normal “calculator”, you get 61.15.