Banff Topology in Dimension 4.5
Questions and Problems

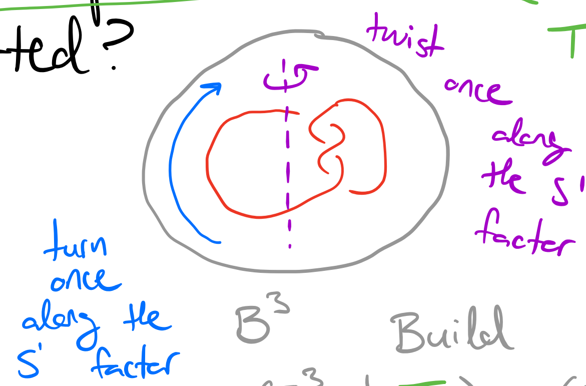
Patrick Naylor

Q Is the Gluck twist of roll spun $7_3 \cong S^4$?

unknotting $\# = 2$ so this is likely the smallest 2-knot for which we don't know the Gluck twist.

Q For any classical knot K, is the turned 1-twisted spun torus of K smoothly unknotted? T

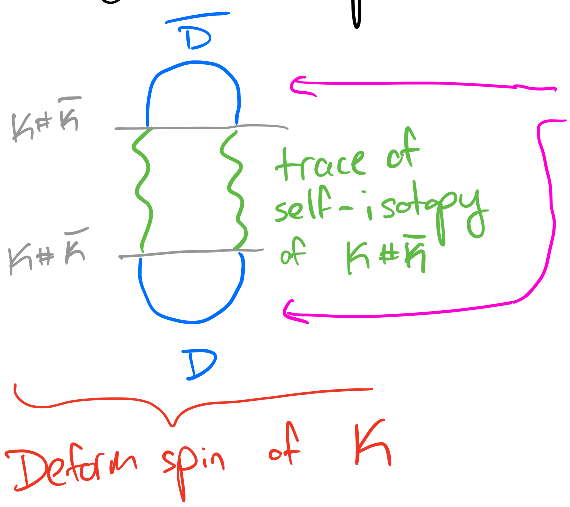
Reasonable since $\pi_1(S^4 \setminus T) \cong \mathbb{Z}$.



Build $(B^3 \times S^1, T) \subset (S^4, T)$

Q What is known about spinning links?

Q Are there similar constructions to deform spins using other ribbon disks?



any point to fixing new $D, \overline{D}$ and varying the choice of self-isotopy of $K\#\overline{K}$?

Anthony Conway

P Find a 2-knot $K \subset S^4$ with trivial Alexander module but nontrivial Rochlin invariant μ.

Q If $K \subset S^4$ has $\pi_1(S^4 \setminus K) \cong \mathbb{Z}$, must K be smoothly unknotted?

True topologically by Freedman.

(This is a well-known question; the affirmative version is called the "smooth unknotting conjecture.")

Q What invariants could help with the previous question? (Also a well-known question.)

Q Can you define μ for nullhomologous 2-knots in other 4-manifolds?

Q What invariants determine the homotopy type of $S^4 \setminus \nu(K)$?

Conjecturally $(\pi_1, \underbrace{\pi_2}_{\text{as a } \mathbb{Z}\pi_1\text{-module}}, k)$. ← "homotopy 2-type"

This has been studied by e.g. Lomonaco.

Arunima Ray

P Define $fq(R, R')$ for more surfaces R and R' (e.g. positive-genus, non-orientable).

P Extend lightbulb theorem (Gabai) to non-orientable ambient 4-manifold.

Q Can the conditions on the dual sphere G in the lightbulb theorem be refined?

- Need $G \cdot G \equiv 0 \mod 2$; see Klug-Miller Ex 7.2.
  What if $G \cdot G$ is even and nonzero?

- Need G to intersect each of R, R' in a single point; see Sato '91 or Klug-Miller Ex 7.3
  What if $G \pitchfork R = G \pitchfork R' = pt$ but G is immersed?

Mark Powell

Q Is every 2-link slice?

P Classify n-component link maps $\bigsqcup^n S^2 \looparrowright S^4$ up to link homotopy for $n \geq 3$ (n=1 trivial; n=2 Schneiderman–Teichner)

Rob Schneiderman

Q Are there settings in which fq can be defined considering unbased homotopies?

e.g. answer is "yes" for $fq(S_0, S_1)$ when there is an immersed sphere $G \looparrowright X^4$ with $G \cdot S_i \equiv 1 \mod 2$.

More details: for H a homotopy between 2-spheres in M^4, have $fq(H) \in \mathbb{F}_2 T$ with $T = \{g \in \pi_1 M \mid g^2 = 1, g \neq 1\}$ the self-intersection invariant.

$fq(H) = \mu(S^2 \times I \to M \times \mathbb{R} \times I) \in \mathbb{Z}\pi_1 M / g + g^{-1}, 1$

on thickened track of H

collapses to $\mathbb{F}_2 T$ since $H \hookrightarrow M \times \{pt\} \times I$

So for $R, R' \subset M$ homotopic 2-spheres, could set

$$fq(R, R') = fq(H) \in \mathbb{F}_2 T \,/\, fq(J)$$

H: homotopy R to R'

J ranges over self-homotopies of R

almost... this "quotient" might involve an affine action.

Problem

If $\exists$ a J with $fq(J) \notin \mu(\pi_3 M)$ then under J a basepoint of R traces out a nontrivial element $s \in \pi_1 M$ of $[R] \in \pi_2 M$ under the action of π_1.

Get indeterminacy in $fq(R, R')$ described by $t \longmapsto fq(J) + sts^{-1}$ $\forall t$ rather than $t \mapsto fq(J) + t$.

- If $fq(J) \in \pi_3 M$ then on $\mathbb{F}_2 T / \mu(\pi_3 M)$ this reduces to conjugation by s.
- If $\operatorname{stab}[R] = 1$ (e.g. R has dual sphere) then $s = 1$.

So finding a J with $fq(J) \in \mu(\pi_3 M)$ or finding more examples of R, M where $fq(J) \in \mu(\pi_3 M)$ $\forall J$ would help in defining target of fq without needing a dual sphere.

Q Does there exist a self-homotopy J of some $S^2 \subset M^4$ such that $\mu(J) \notin \mu(\pi_3 M)$?

$\left(\mu(J) = \sum_{C \subset J \cdot J} g_C \in \mathbb{Z}\pi_1 M / g + g^{-1}, 1 \right)$

sheet-changing loop of C represents $g_C \in \pi_1 M$

Answer is "no" if $[S^2] \in \pi_2 M$ has a trivial stabilizer in $\pi_1 M$

e.g. if there is an immersed sphere G with $S^2 \cdot G \equiv 1$.

Ryan Budney

Q Do barbells generate $\pi_0 \mathrm{Diff}(D^4)$? Or $\pi_0 \mathrm{Diff}(S^1 \times D^3)$?

$\Theta_2 : S^1 \times D^3 \circlearrowleft$ see Budney-Gabai

Q Is Θ_2 nontrivial? (Budney-Gabai show Θ_n nontrivial for $n > 3$.)

Q Can Watanabe's invariants be described in terms of scanning?

Q Does knotting of barbells matter?

← like if these spheres are knotted, does that give anything interesting to study?

Q Are the Hatcher-Wagoner invariants surjective in dimension 4?

P Understand $\mathrm{Diff}(S^2 \times S^2)$ or $\mathrm{Diff}(\mathbb{CP}^2)$. Can you find generators of π_0?

Q What is the difference between Diff(spin 4-mfd) and Diff(non-spin 4-mfd)?

Q What is $\pi_0 \mathrm{Diff}(D^4)$?

Slava Krushkal

Q Barbells generate the subgroup of $\pi_0 \mathrm{Diff}(S^1 \times B^{n-1})$ that is null in pseudoisotopy for $n \geq 6$ (Hatcher-Wagoner). Does this hold for $n = 5$ too?

P Find null-pseudoisotopies for the Budney-Gabai diffeomorphisms of $S^1 \times D^3$ and compute their Hatcher-Wagoner obstructions.

Q Budney-Gabai proved $\pi_0 \mathrm{Diff}(S^1 \times B^3, \partial)$ contains an infinite set of linearly independent elements. Are (some of) these elements still non-trivial up to topological isotopy?

Dave Auckly

P Compare $\pi_n(\mathrm{Diff}(Z, D^4))$ to $\pi_n(\mathrm{Homeo}(Z, D^4))$ up to stabilizing Z^4 by $S^2 \times S^2$.

Tadayuki Watanabe

Q Do the graph classes in $\pi_k B\mathrm{Diff}(D^4)$ survive under the map $\pi_k B\mathrm{Diff}_\partial(D^4) \to \pi_k B\mathrm{Diff}_\partial(D^4 \# (S^2 \times S^2))$?

(From the Weiss fiber sequence)

Q Are the theta-graph (or barbell) classes mapped to nontrivial elements by $\pi_1 B\mathrm{Diff}_\partial(D^3 \times S^1) \to \pi_1 B\mathrm{Homeo}_\partial(D^3 \times S^1)$?

- Yes for $\pi_{d-3} B\mathrm{Diff}_\partial(D^{d-1} \times S^1)$, $d \geq 5$.
- Yes for $\to \pi_1 BPL_\partial(D^3 \times S^1)$.

Q Can a configuration space integral invariant be defined on $\pi_1 B\mathrm{Homeo}_\partial(D^3 \times S^1)$?

Are invariants of topological embeddings $\tilde{\Delta}_X \hookrightarrow \widetilde{X \times X}$ helpful?

What about in $\pi_{d-3} B\mathrm{Homeo}_\partial(D^{d-1} \times S^1)$?

Q What is the image of

$$p : \pi_1 \mathcal{M}^{psc}_\partial(X^4) \to \pi_1 B\mathrm{Diff}_\partial(X^4)?$$

($\mathcal{M}^{psc}_\partial(X^4)$: moduli space of positive scalar curvature metrics.)

- Classes detected by Seiberg-Witten theory are not in $\mathrm{Im}\, p$ (don't admit fiberwise psc metrics)
- Graph classes are in $\mathrm{Im}\, p$ (Botvinnik-W 2021)

compare to $\mathcal{M}_{GL}(X^4) \subset \pi_0 \mathrm{Diff}_\partial(X^4)$ (the subgroup for which parameterized Gromov-Lawson construction works; see Botvinnick-Hanke-Schick-Walsh)

Gay-Hartman 2022: $\mathcal{M}_{GL}(D^4) \cong \mathbb{Z}/2$ or 0.