Topology in Dimension 4.5 – Session C
Recent Results

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Recent Results

Theorem: (Watanabe) $\pi_1 \mathrm{Diff}(D^4)$ is non-trivial.

In this presentation $\mathrm{Diff}(M)$ denotes all diffeomorphisms of M that restrict to the identity on ∂M.

Recent Results

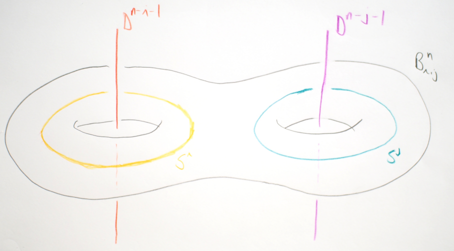
Theorem: (B,G) $\pi_0 \mathrm{Diff}(S^1 \times D^3)$ is not finitely-generated.

B,G techniques

Construct essential family $\Omega^{n-j}S^i \to \mathrm{Diff}(\mathcal{B}^n_{i,j})$, for any $1 \leq i \leq j < n$.

$\mathcal{B}^n_{i,j}$ is the boundary connect-sum of $S^i \times D^{n-i}$ and $S^j \times D^{n-j}$, called a **barbell manifold**.

$$\mathcal{B}^n_{i,j} \equiv S^i \times D^{n-i} \natural S^j \times D^{n-j}$$



Notice the first non-trivial homotopy-group of $\Omega^{n-j}S^i$ is in dimension $i+j-n$, and it is $\mathbb{Z}$, provided $i+j \geq n$.

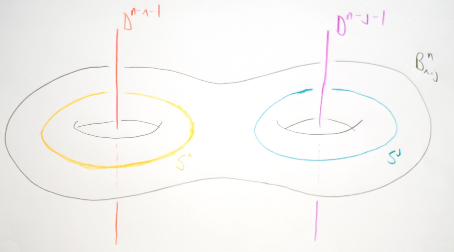
B,G techniques

Strategy is to find suitable embedding $\mathcal{B}^n_{i,j} \to N^n$ and extend the family, giving a potentially essential map $\Omega^{n-j} S^i \to \mathrm{Diff}(N)$.

Restriction: The maps

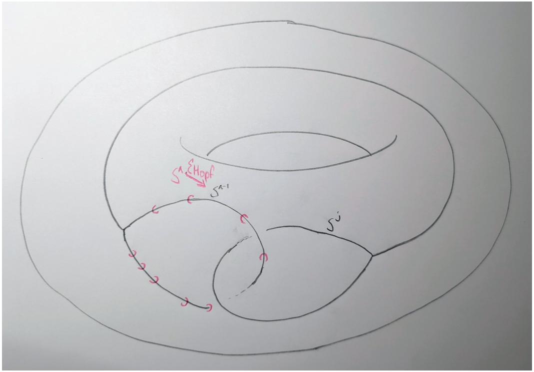
$$\mathrm{Diff}(\mathcal{B}^n_{i,j}) \to \mathrm{Diff}(S^i \times D^{n-i}) \qquad \mathrm{Diff}(\mathcal{B}^n_{i,j}) \to \mathrm{Diff}(S^j \times D^{n-j})$$

obtained by filling the j-handle (or i-handle respectively) are null when restricted to our barbell family, thus suitable embeddings $\mathcal{B}^n_{i,j} \to N$ need to be ‘linked’.



Recent Results - High dimensions

In dimensions $n \geq 6$ we use the 'Hopf implantation' of the barbell $\mathcal{B}^n_{i,j}$ with $i + j = n$ in $S^1 \times D^{n-1}$.

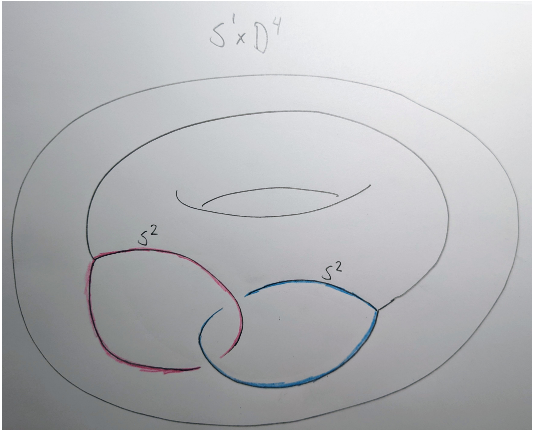


$$\pi_{n-j} S^i = \pi_i S^i \equiv \mathbb{Z} \to \pi_0 \mathrm{Diff}(S^1 \times D^{n-1})$$

These *recover** the Hatcher-Wagoner diffeomorphisms of $S^1 \times D^{n-1}$.

Recent Results - High dimensions

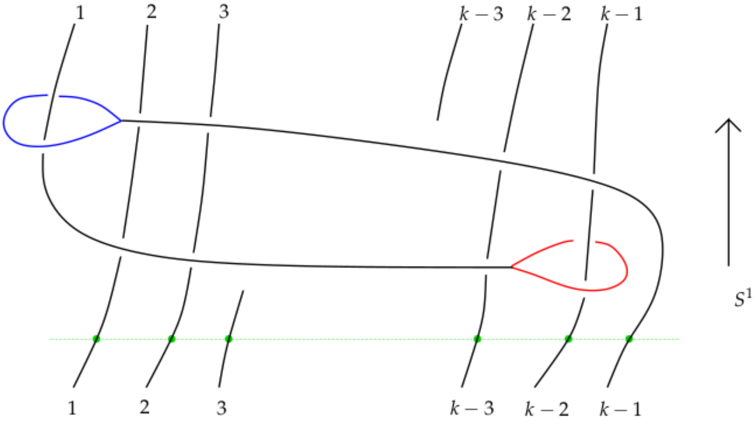
In dimension $n = 5$ we use the 'simple linking' embedding of $\mathcal{B}^5_{2,2}$.



$$\pi_{5-2}S^2 = \pi_3 S^2 \equiv \mathbb{Z} \to \pi_0 \mathrm{Diff}(S^1 \times D^4)$$

Recent Results - High dimensions

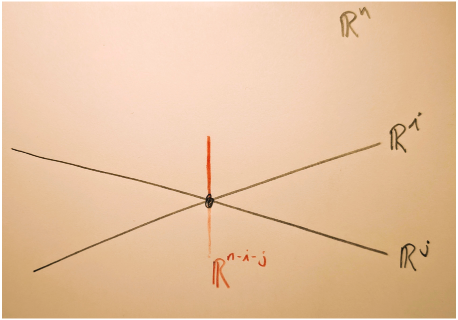
In dimension $n = 4$ we use the 'handcuff embeddings' of $\mathcal{B}^4_{2,2}$.



δ_k barbell in $S^1 \times D^{n-1}$

Def'n of barbell diffeomorphism

Consider linearly embedded copies of $\mathbb{R}^i$ and $\mathbb{R}^j$ in $\mathbb{R}^n$, intersecting in a point.



Provided $n - i - j > 0$ we can perturb-away the intersection.
Provided $n - i - j > 1$, all such perturbations are isotopic.

Def'n of barbell diffeomorphism – Paint Mixing ODE

Digression: $\Omega^{n-j}S^i \to \mathrm{Diff}(\mathcal{B}^n_{i,j})$

Def'n of barbell diffeomorphism – Paint Mixing ODE

Theorem: (Palais) If M is a manifold, and N a submanifold, the restriction map

$$\mathrm{Diff}(M) \to \mathrm{Emb}(N, M)$$

is a locally-trivial fibre-bundle.

That the map is a Serre fibration is due to Cerf. The proof is essentially the isotopy-extension theorem 'with parameters'.

The consequence we need is the homotopy long-exact sequence

$$\cdots \to \pi_k \mathrm{Diff}(M) \to \pi_k \mathrm{Emb}(N, M) \to \pi_{k-1} \mathrm{Diff}(M \text{ fix } N) \to \cdots$$

specifically the red arrow.

Def'n of barbell diffeomorphism

Consider our map $S^{n-i-j-1} \to \mathrm{Emb}(\mathbb{R}^i \sqcup \mathbb{R}^j, \mathbb{R}^n)$.

The induced element of $\pi_{n-i-j-2}\mathrm{Diff}(\mathbb{R}^n)$ is supported in a ball containing the original double-point.

Given that we can thicken the copies of $\mathbb{R}^i$ and $\mathbb{R}^j$, the diffeomorphism family can be chosen to be the identity on these thickened copies. Thus we have an element of

$$\pi_{n-i-j-2}\mathrm{Diff}(\mathcal{B}^n_{n-i-1,n-j-1})$$

Let $i' = n - i - 1, j' = n - j - 1$ then this is an element of

$$\pi_{i'+j'-n}\mathrm{Diff}(\mathcal{B}^n_{i',j'})$$

This is the barbell diffeomorphism family, on the first non-trivial homotopy group of $\Omega^{n-j'} S^{i'}$.

Case $i+j=n$, $\pi_i S^i \to \pi_0 \mathrm{Diff}(\mathcal{B}^n_{i,j})$