Dehn surgery on links + the Thurston norm

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Background

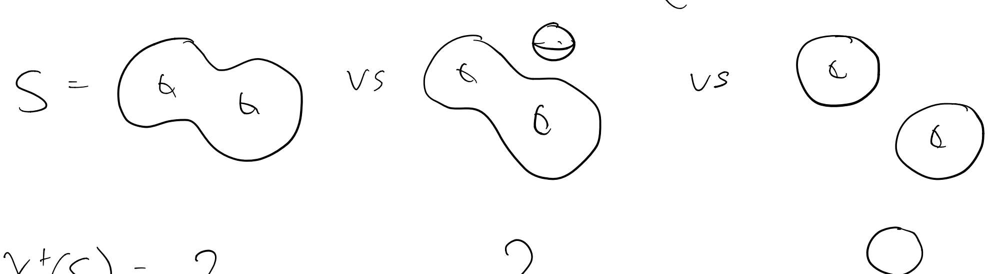
Thurston norm:

pseudonorm on (regular or rel ∂) homology of 3-mfd (cpct, smooth)

$$x : H_2 = H_2(M, \partial M ; \mathbb{R}) \longrightarrow \mathbb{R}^{\geq 0}$$

$\alpha \in H_2$ integral $\Rightarrow$ α rep'd by embedded surface in M

$$x(\alpha) = \min_{[S]=\alpha} \chi^+(S) \text{ where } \chi^+(S) = \begin{cases} \max\{0, -\chi(S)\} & S \text{ connected} \\ \sum_{i=1}^{n} \chi^+(S_i) & S = S_1 \cup \cdots \cup S_n \end{cases}$$



$\chi^+(S) = 2$ $\qquad$ 2

$\beta \in H_2 \quad p\beta = \alpha$ rational

$\rightsquigarrow x(\beta) = \frac{1}{p} x(\alpha) \in \mathbb{Q}$

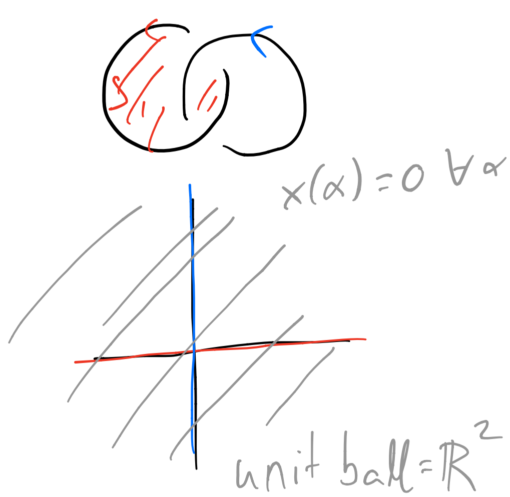
use approximation to extend to real classes.

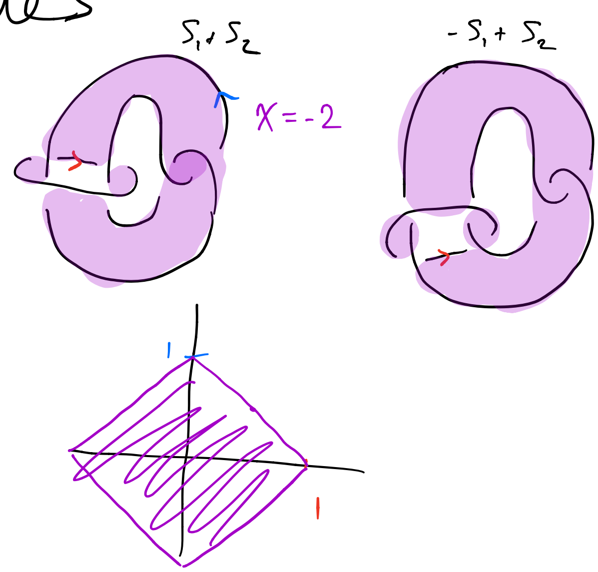
Properties (Thurston)

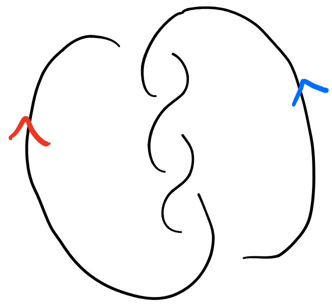
- If $x(\alpha) = 0 \Leftrightarrow \alpha = 0$, then x a norm
- Unit ball $B_x \subset H_2(M, \partial M; \mathbb{R}) \cong \mathbb{R}^n$

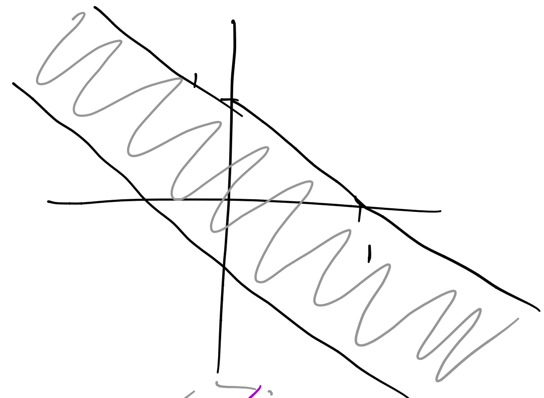
convex polyhedron symmetric about origin
boundaries integral affine equations

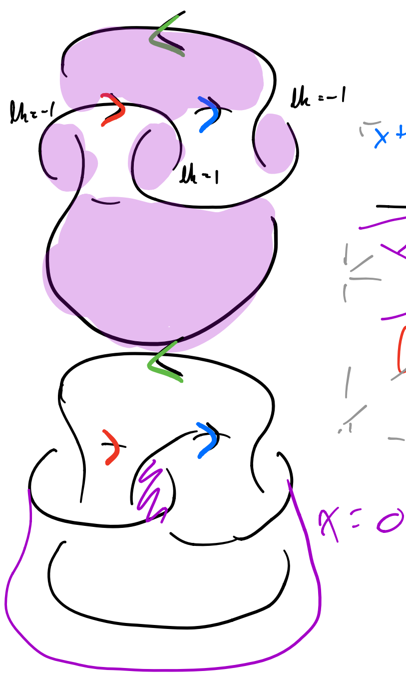
Better examples

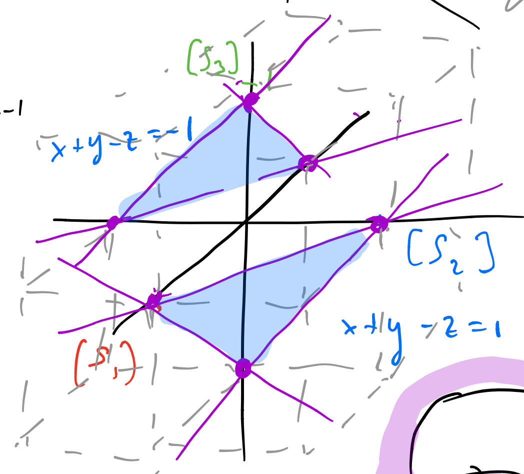


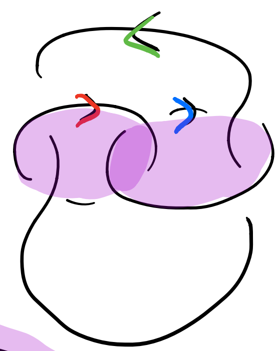




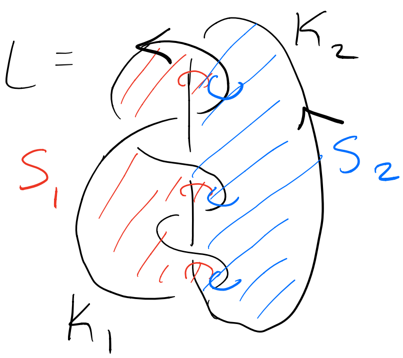




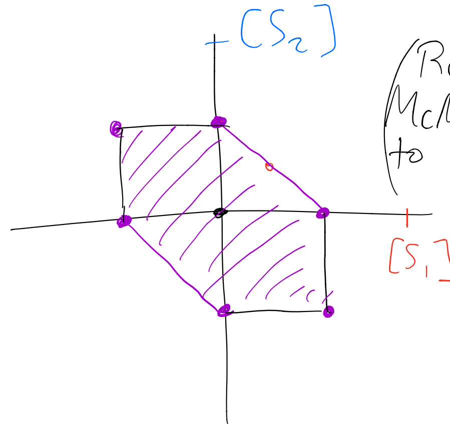




<u>Ex</u> $M = S^3 - \mathring{N}(L)$ $\quad H_2(M, \partial M; \mathbb{R}) \cong \mathbb{R}^2$



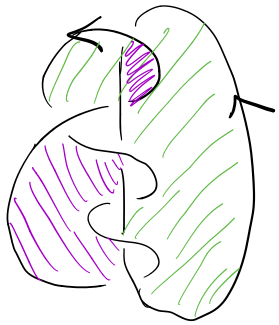
$\chi_+(S_1) = 2 \quad \chi_+(S_2) = 2$



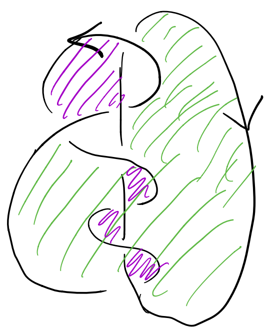
(Refer to McMullen to complete ball)

B_x

$[S_1]+[S_2]$ $\qquad$ $[S_1]-[S_2]$



$\chi^+=4$



$\chi^+=2$

Detecting Thurston norm

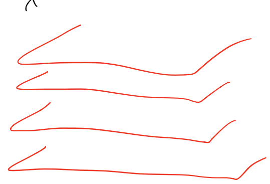
Rmk: $\Rightarrow [S]$ primitive

Thurston: If S connected cpct surface + leaf of taut foliation, then S norm-minimizing ($[S]\neq 0$, if S' connected with $[S']=[S]$, then $\chi(S)\geq\chi(S')$ $\Rightarrow \chi^+(S)=x([S])$)

taut foliation of 3-mfd M is

$M=\bigsqcup_\lambda L_\lambda$ each L_λ oriented surface,

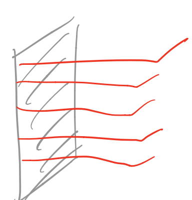
and locally



$L_\lambda=\{z=\lambda\}$

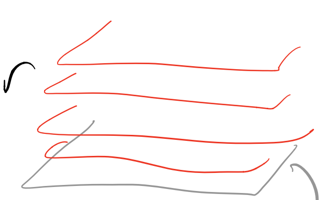
or

∂



$L_\lambda=\{z=\lambda\}$
$M=\{y\geq 0\}$

or

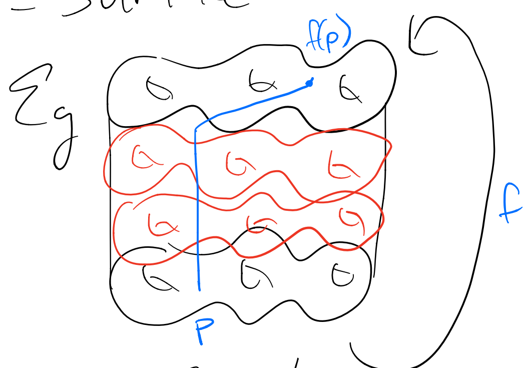


∂

$L_\lambda=\{z=\lambda\}$
$M=\{z\geq 0\}$

Foliation $\mathcal{F} = \{L_\lambda\}$ <u>taut</u> if $\exists$ prop embedded 1-mfd in M which meets <u>every</u> L_λ, <u>always</u> transversely.

<u>Ex.</u> $M =$ surface bundle over S^1



Taut foliation where $L_\lambda = \Sigma_g \times \lambda$

$\rightsquigarrow$ $\Sigma_g \times pt$ norm-minimizing

$$\chi([\Sigma_g]) = \begin{cases} 2g-2 & g>0 \\ 0 & \Sigma_g = S^2 \end{cases}$$

Cor (Gabai)

Property R: If $S^3_0(K) \cong S^2 \times S^1$, then $K = \text{unknot}$

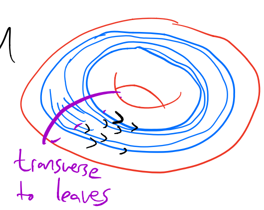
Thm (Gabai)

If $M = $ cpct, connected, irreducible, orientable 3-mfd with $\partial M = \sqcup T^2$ and $S \subset M$ connected norm-min surface, then S is leaf of taut foliation.

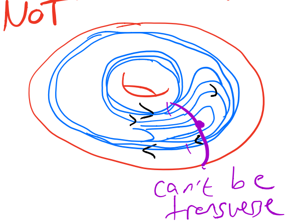
If ∂S only one orientation on each ∂M, then ∂ foliation is Reebless

"properly norm-min"

(i.e. ∂M



NOT



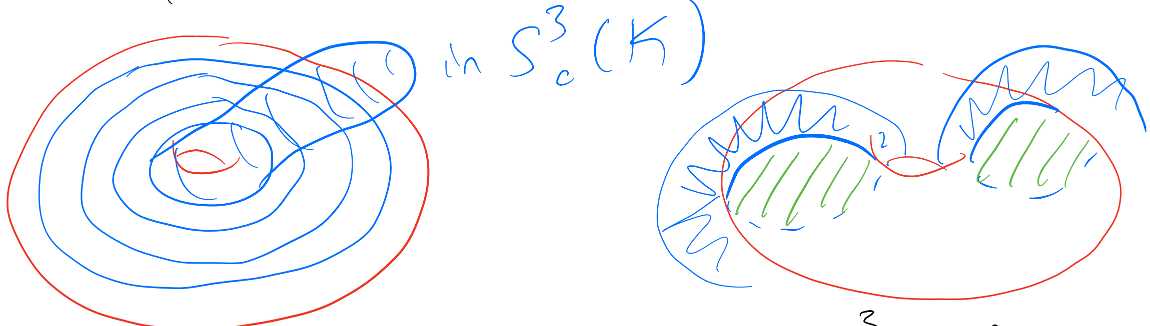
If $M = S^3 - \mathring{N}(\text{knot})$ then ∂ foliation only compact circles. $(*)$

Proof of Property R

K = knot in S^3

S = min-genus Seifert surface for K

Thm $\Rightarrow$ S leaf of taut foliation on $M = S^3 - \mathring{N}(K)$, ∂ foliation = circles



Dehn fill M to get $S^3_0(K)$.
Cap off each circle with disk to get taut foliation including $\hat{S}$ as leaf. $\rightsquigarrow$ $\hat{S}$ norm-min

If $S^3_0(K) = S^2 \times S^1$, then $[\hat{S}] = [S^2 \times pt]$
so $\hat{S} \cong S^2 \Rightarrow S \cong D^2 \Rightarrow K = \text{unknot}$.

(*) Not true for links

$S \cong T^2 \setminus D^2$
norm-minimizing

$\chi^+(S) = 1$

In $S^3_{\partial S}(L)$
$= S^3_{(0,\infty)}(K_1, K_2)$
$\cong S^2 \times S^1$

$g(\hat{S}) = 1$
$[\hat{S}] = [S^2 \times pt]$
$\therefore$ $\hat{S}$ not norm-minimizing

rational homology sphere

To state about H_2 instead of space of slopes

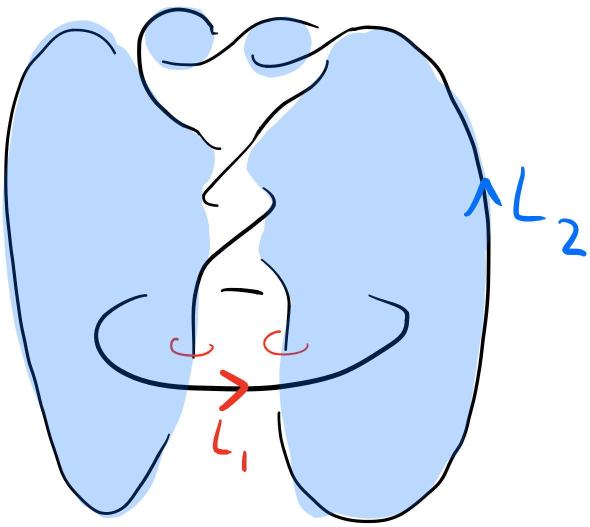
$M := S^3 - \mathring{N}(L)$
$H_2 := H_2(M, \partial M; \mathbb{R})$
norm $x : H_2 \to \mathbb{R}^{\geq 0}$
non-degenerate

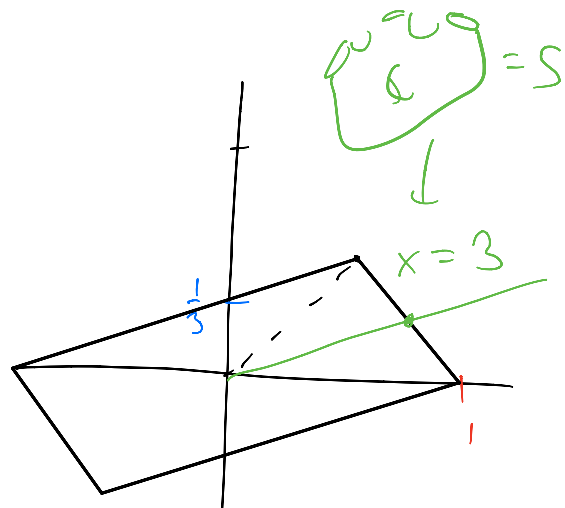
Thm If $L = \underbrace{K_1 \sqcup K_2}_{(n>1)\text{-component}} \subset S^3$, $lk(K_1, K_2) \neq 0$,
then $\exists$ finite $E \subset H_2$ so if
primitive $\alpha \in H_2 - E$ and $[S] = \alpha$, S norm-min
dimension $(n-2)$

THEN $\hat{S}$ norm-min in $S^3_{\partial S}(L)$

- because $\hat{S}$ leaf of taut foliation
- because S leaf of taut foliation w/ ∂ cpct circles

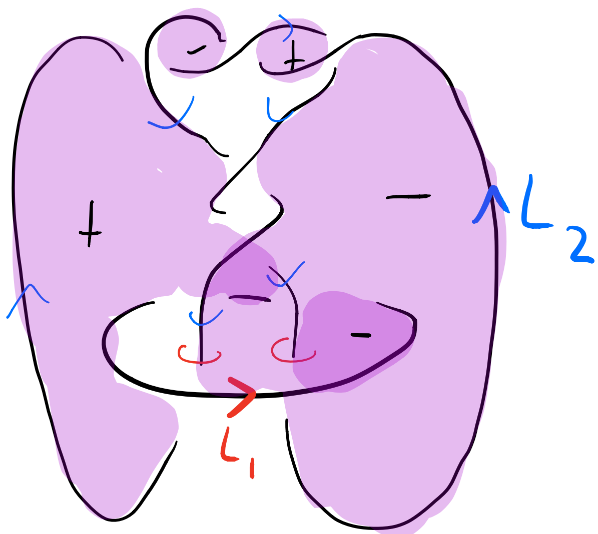
$lk = -2$

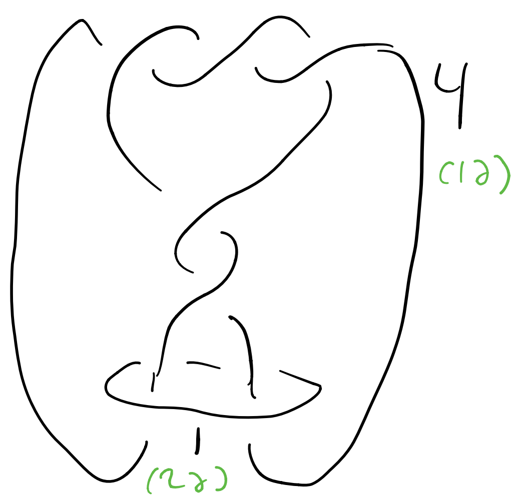




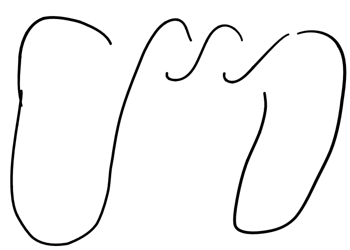
$2S_1 + S_2$

$$\partial = 2\lambda_1 + \lambda_2 + 4\mu_2 + 2\mu_1$$

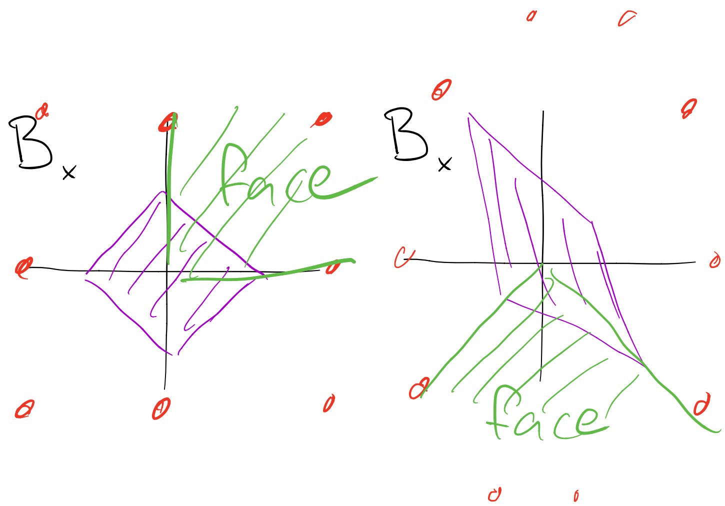




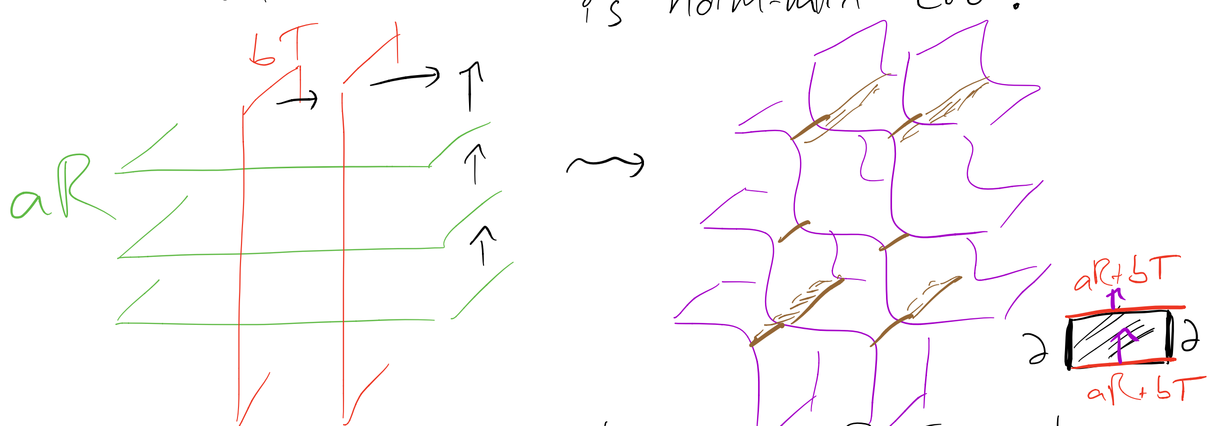
$\chi = 5 - 7 = -2$

$=$  $= S^1 \times S^2$

$E \supset$ min-genus primitive elements of each face + vertices
different from X



faces useful because in one face, x is linear. So if R, T norm-min for adjacent vertices of B_x, then $aR + bT$ (cut-and-paste) is norm-min too.



For each arc in $R \cap T$, get ab product disks

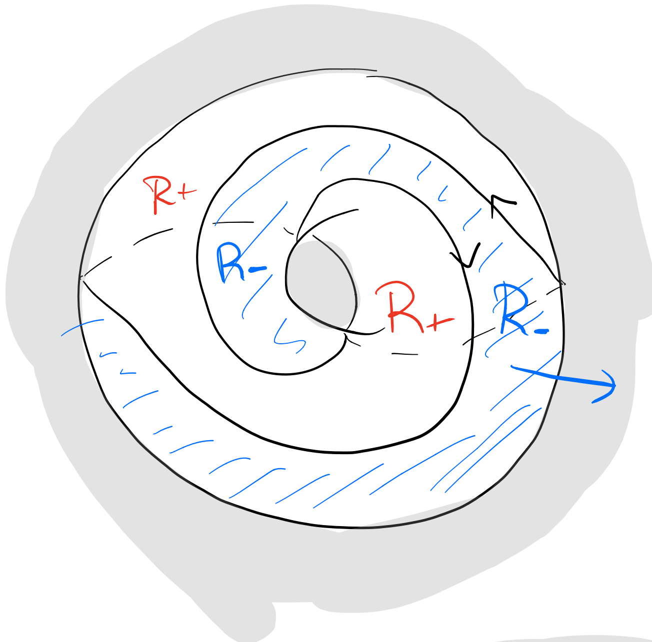
Gabai: Sutured manifolds

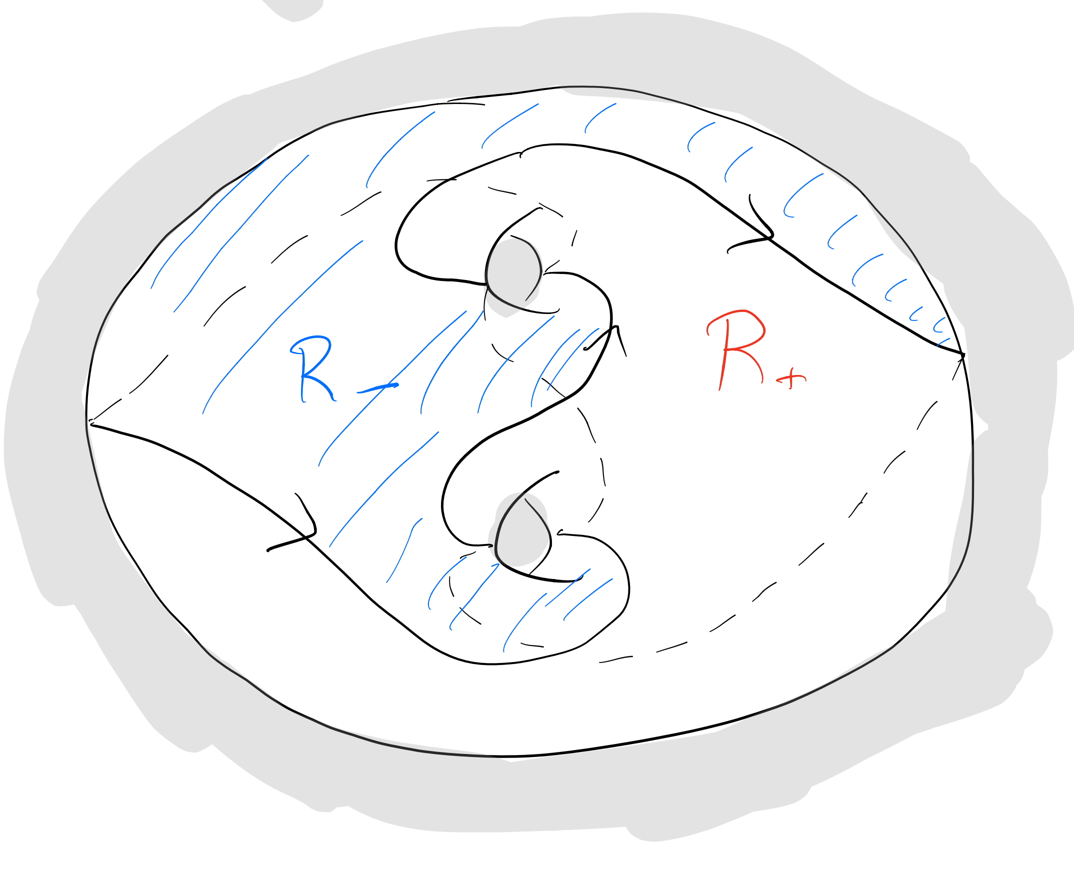
(M, γ) sutured manifold is
cpct 3-mfd M with
disjoint annuli $A(\gamma)$, tori $T(\gamma)$ $\gamma \subset \partial M$
s.t.
- each annulus in $A(\gamma)$ has oriented core (called sutures $s(\gamma)$)
- components of $\partial M - \gamma$ oriented

$$\partial M - \gamma = \underbrace{R_+(\gamma)}_{\text{normal out of } M} \sqcup \underbrace{R_-(\gamma)}_{\text{normal into } M}$$

so orientations on $\partial M - \gamma$ induced by $s(\gamma)$.

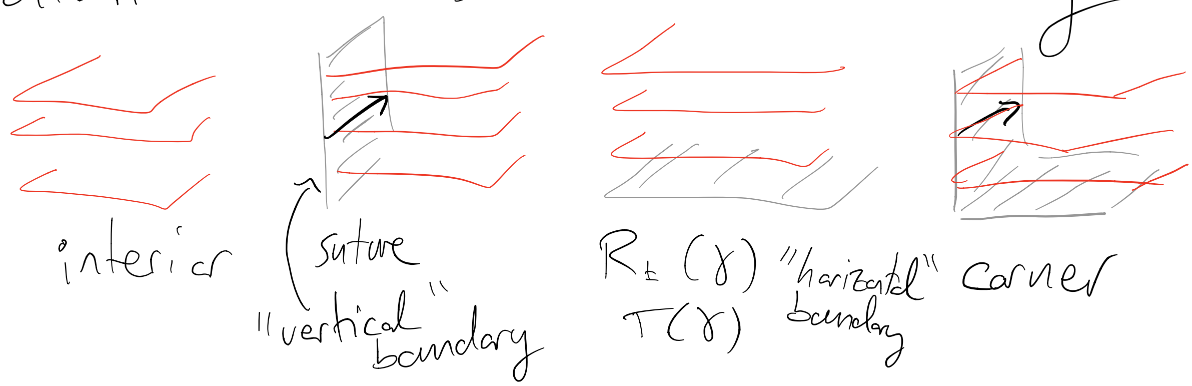
Ex



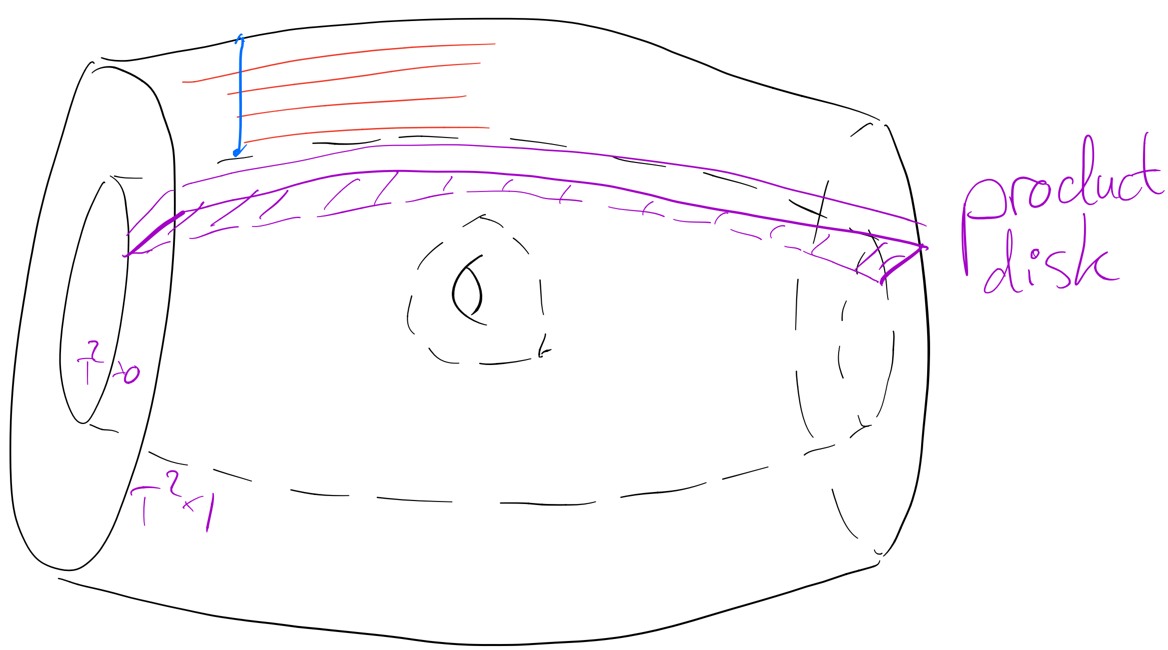


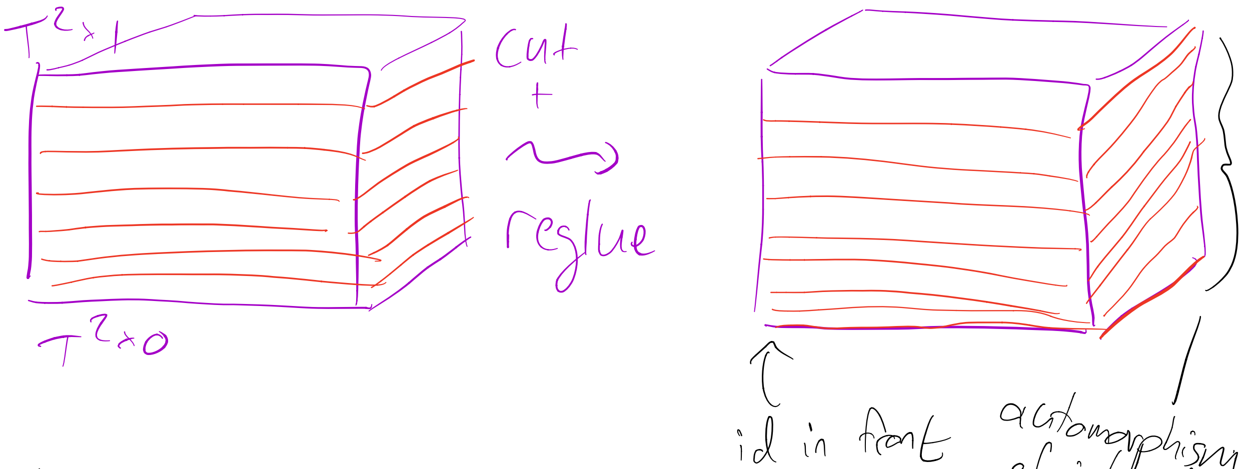
Sutured mfd is <u>taut</u> iff $R_+(\gamma)$ and $R_-(\gamma)$ norm-minimizing.

Foliation on sutured mfd locally



$$M_g = (T^2 \setminus 2\mathring{D}^2) \times I \qquad L_\lambda = T^2 \setminus \mathring{D}^2 \times pt$$

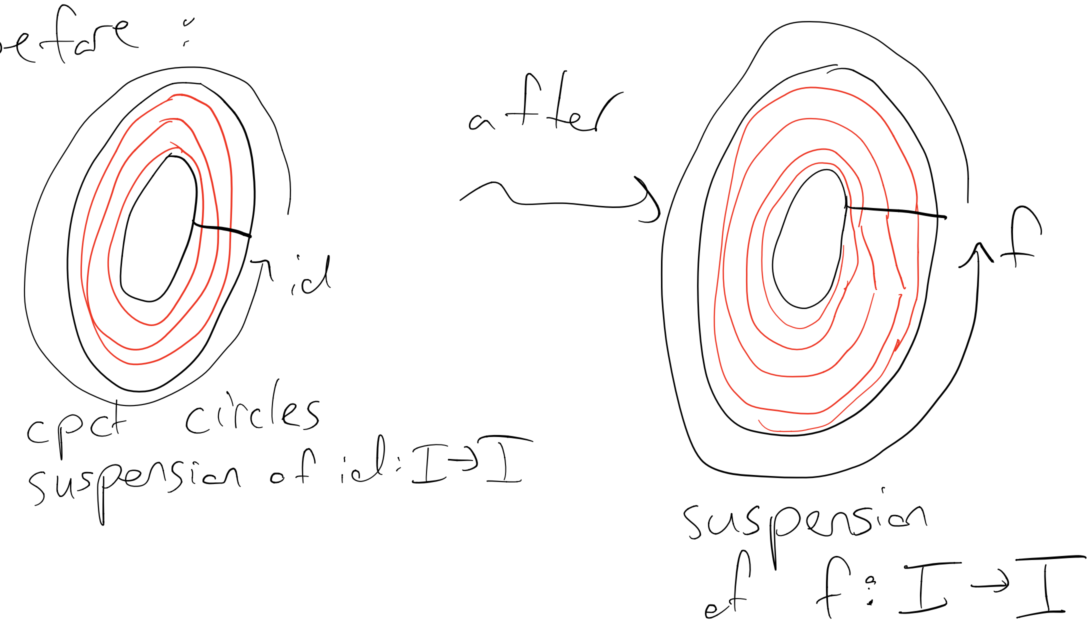




Now a lot of non-compact leaves, but still taut.

Effect on ∂

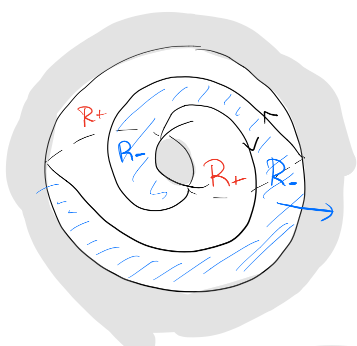
Before:



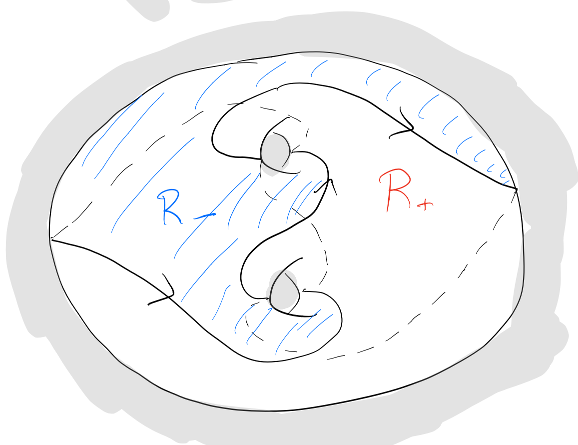
Complementary sutured mfd:

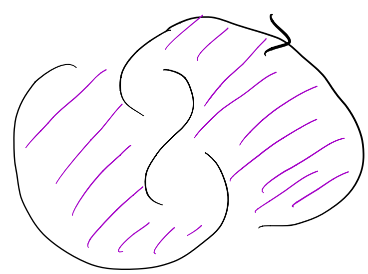
$$\underset{\text{surface}}{S} \subset N^3$$

$$\rightsquigarrow \underbrace{(N^3 \setminus i(S), \overbrace{\partial S \times I}^{\text{annuli}})}_{\text{sutured mfd.}}$$

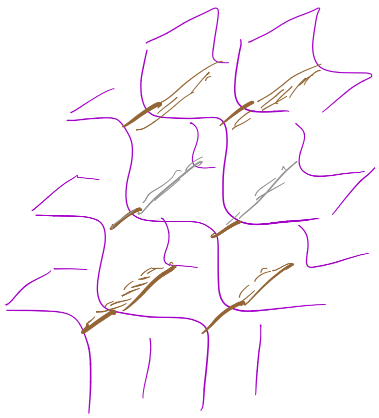


complement of  $\subset S^3$



 $\subset S^3$

taut $\Longleftrightarrow$ S norm-minimizing



So in complementary sutured mfd to $aR+bT$, we find many product disks

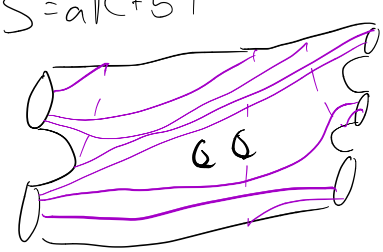
$aR+bT$ From Gabai's thm, have ∂reebless taut foliation on $S^3 \setminus \nu(L)$ achieving $S = aR+bT$ as a leaf.

Can use product disks to change ∂ foliation.

Main observation:

Can simultaneously make ∂ foliation on both components just compact circles unless ∂_+(product disks) cuts S into disks and one genus-$g(S)$ surface.

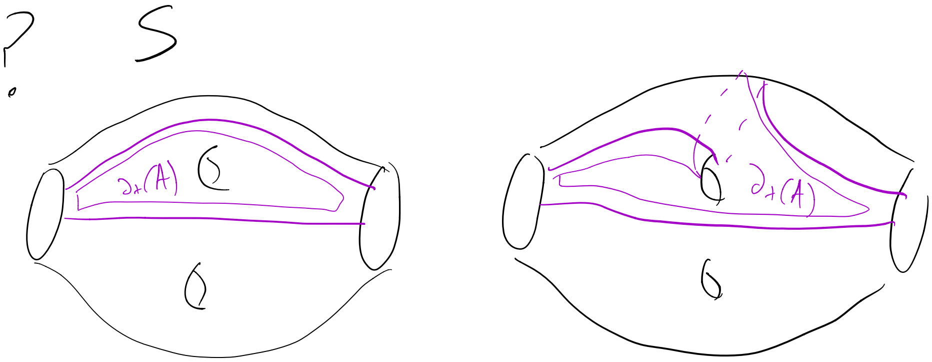
$S = aR+bT$



The genus-$g(S)$ piece is then in $R+T$.

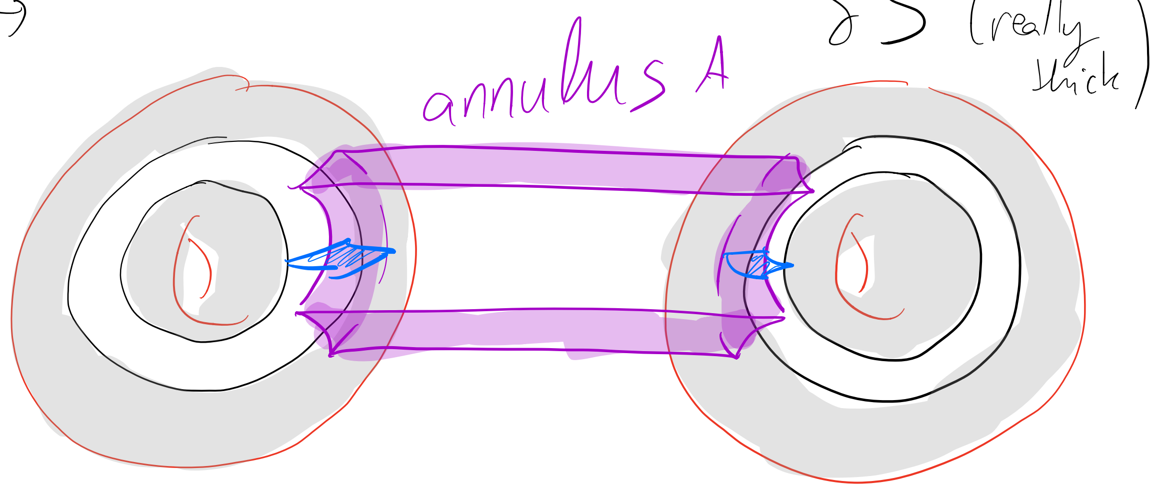
$\Rightarrow g(xR+yT) \geq g(S)$ $\forall x,y \geq 1$.

Why?



annulus in complement to S

⤳

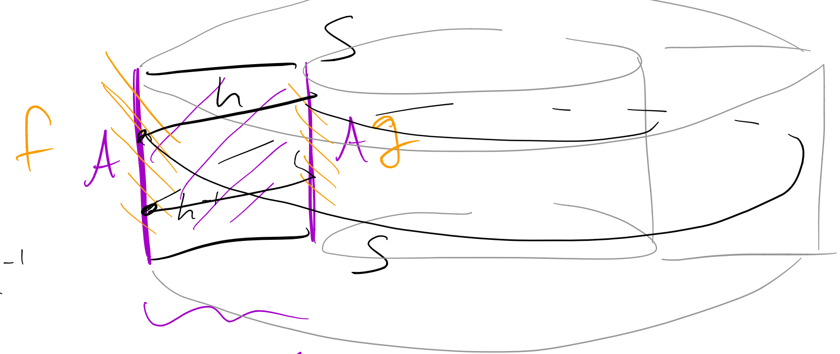


∂M

Delete $A \times I$ from M to make new torus ∂ component

Use product disks to make foliation ∂ on $S^3 \setminus \nu(L)$ cpct circles, at cost of foliation on new ∂ being bad.

Want to glue $A \times I$ back



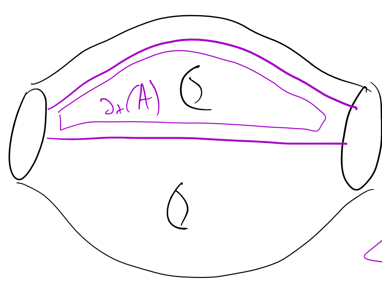
$f = hgh^{-1}$

$A \times I$

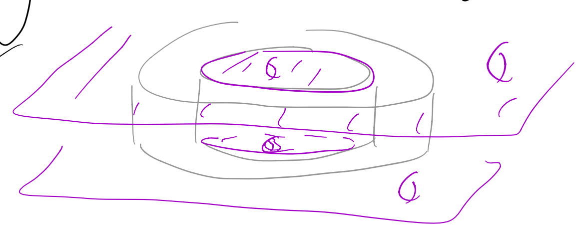
$f, g : I \to I$

Can glue back in solid torus if f, g conjugate

Case 1: $\partial_+ A$ separating on S_+



(into two + genus pieces)



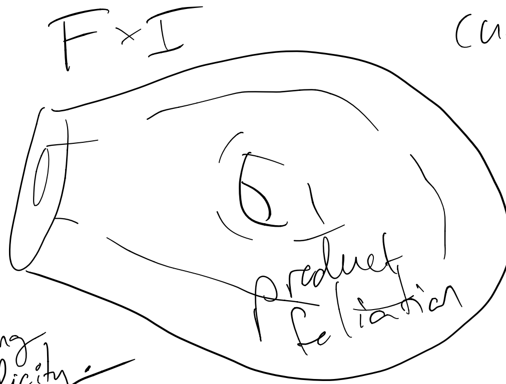
Lemma (Gabai)

If F pos genus surface, can find taut foliation on F achieving any suspension on $\partial F \times I$

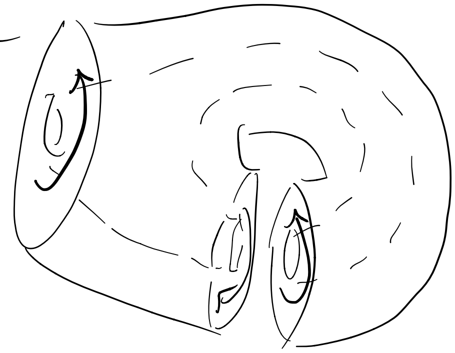
Pf

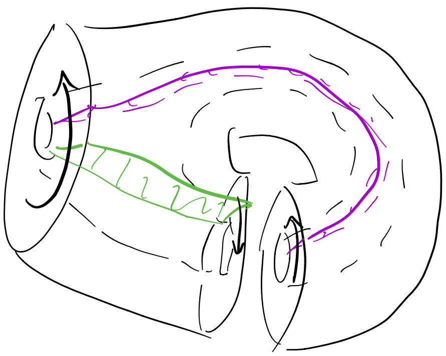
Fix $\varphi: I \to I$

assume φ expanding for simplicity.



cut essential curve $\rightsquigarrow$





change suspension $\rightsquigarrow$

$\mu_1: I \to I$ on green

$\mu_2: I \to I$ on purple

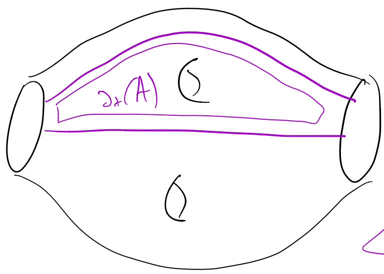
product disks

If μ_1^{-1} conjugate to μ_2, can reglue deleted annulus $\times I$

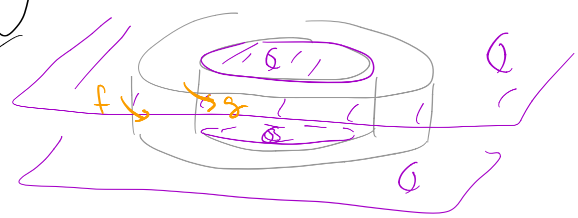
So take μ_1 contracting, $\mu_2 = \varphi\mu_1^{-1}$ expanding

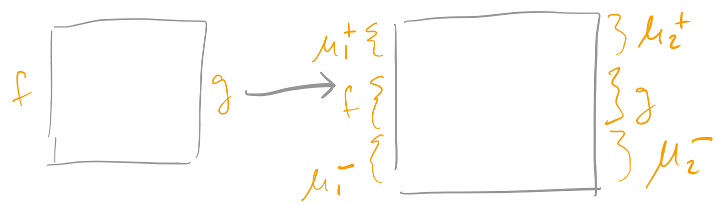
$\rightsquigarrow \mu_2, \mu_1$ conjugate, $\mu_2\mu_1 = \varphi$.

Case 1: $\partial_+ A$ separating on S_+



(into two + genus pieces)



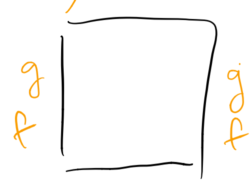


free choice of $\mu_i^\pm$

So take $\mu_1^+ = g$ $\quad \mu_2^- = f$

$\mu_1^- = \mu_2^+ = id$

$\rightsquigarrow$

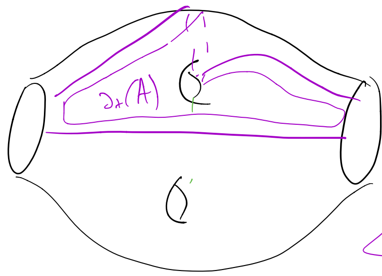


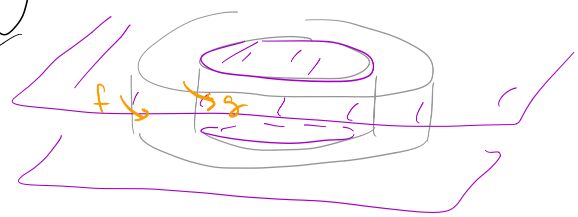
reglue annulus

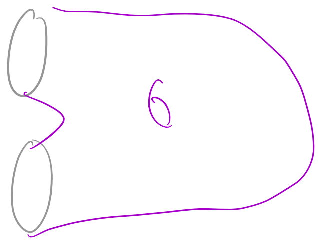
$\rightsquigarrow$ taut foliation of complementary sutured mfd to S with $\partial =$ circles

$\rightsquigarrow$ taut foliation of $S^3 \setminus \nu(L)$ achieving S as leaf with $\partial =$ circles

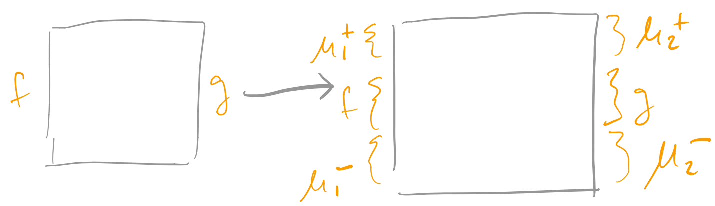
Case 2: $\partial_+ A$ nonseparating on S_+







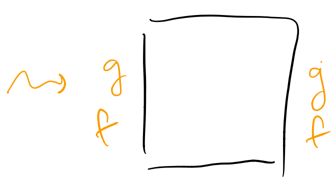
Use lemma to freely change suspensions near S



free choice of $\mu_i^\pm$

So take $\mu_1^+ = g \quad \mu_2^- = f$

$\mu_1^- = \mu_2^+ = id$



reglue annulus

$\rightsquigarrow$ taut foliation of complementary sutured mfd to S with $\partial =$ circles

$\rightsquigarrow$ taut foliation of $S^3 \setminus \nu(L)$ achieving S as leaf with $\partial =$ circles