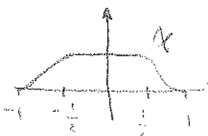
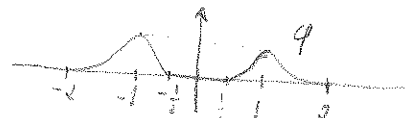
Preliminaries and general tools

LP decomposition: Let $\chi \in C_0^\infty(B_1(0))$ be such that $\chi|_{\overline{B_{\frac{1}{2}}(0)}} = 1$ and even and define



$$\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$$



and denote $\lambda_q := 2^q$. We obtain a partition of unity:

$$\chi(\xi) + \sum_{q \geq 0} \varphi(\lambda_q^{-1}\xi) = 1.$$

Furthermore, $|p-q| \geq 2 \Rightarrow \operatorname{supp} \varphi(\lambda_q^{-1}\cdot) \cap \operatorname{supp} \varphi(\lambda_p^{-1}\cdot) = \emptyset$.

Let the FT be def. as $\mathcal{F}u(k) := \int e^{ikx} u(x)\,dx$ and introduce the LP projections

$$\Delta_q u := \begin{cases} \mathcal{F}^{-1}(\varphi(\lambda_q^{-1}\cdot)\mathcal{F}u) = \lambda_q^3\, h(\lambda_q \cdot) * u &, q \geq 0 \\ \mathcal{F}^{-1}(\chi \mathcal{F}u) = \tilde{h} * u &, q = -1 \end{cases},$$

where we define $h := \mathcal{F}^{-1}\varphi$, $\tilde{h} := \mathcal{F}^{-1}\chi$. The LP projection up to the Q^{th} shell is given by

$$S_Q u := \sum_{q=-1}^{Q} \Delta_q u = \mathcal{F}^{-1}(\chi(\lambda_{Q+1}^{-1}\cdot)\mathcal{F}u) = \lambda_{Q+1}^3\, \tilde{h}(\lambda_{Q+1}\cdot) * u.$$

Note that we have

$$u = \lim_{Q \to \infty} S_Q u \text{ a.e.}$$

Def. (Besov space) Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. Then we define the inhomogeneous Besov space

$$B^s_{p,r} := \left\{ u \in \mathcal{S}' \;\middle|\; \|u\|_{B^s_{p,r}} := \|\Delta_{-1} u\|_{L^p} + \left\| \left(\lambda_q^s \|\Delta_q u\|_{L^p}\right)_{q \geq 0} \right\|_{\ell^r} \right\}.$$

Useful inequalities:

(i) $b \geq a \geq 1$, $\overbrace{\alpha \in \mathbb{N}_0^3}^{\text{also fractional derivatives!}}$: $\|D^\alpha \Delta_q u\|_b \leq \lambda_q^{|\alpha| + d(\frac{1}{a} - \frac{1}{b})} \|\Delta_q u\|_a$ (Bernstein inequ. (BI))

(i') $\lambda^{|\alpha|} \|\Delta_q u\|_a \sim \|D^\alpha \Delta_q u\|_a$

(ii) $\|u\|_p \sim \left\| \|(\Delta_q u)_{q \geq -1}\|_{\ell^2} \right\|_{L^p}$ (Littlewood-Paley inequ. (LPI))

(iii) $\infty > p \geq q \geq 1$: $\left[\int_{S_2} \left(\int_{S_1} |F(x,y)|^q \mu_1(dx) \right)^{p/q} \mu_2(dy) \right]^{\frac{1}{p}} \overset{\text{Jensen}}{\leq} \left[\int_{S_1} \left(\int_{S_2} |F(x,y)|^p \mu_2(dy) \right)^{\frac{q}{p}} \mu_1(dx) \right]^{\frac{1}{q}}$ (MI)

In addition, we have the following continuous embeddings for $b \geq a \geq 1$
in the sense of norm-estimates

$$\mathcal{B}^{s}_{a,r} \subseteq \mathcal{B}^{s-d\left(\frac{1}{a}-\frac{1}{b}\right)}_{b,r}$$

$$\mathcal{B}^{0}_{a,2} \subseteq L^{a} \quad \text{for } a \geq 2$$

and furthermore

$$H^{5/6}(\mathbb{R}^3) \subseteq \mathcal{B}^{\frac{2}{3}}_{\frac{9}{4},2}(\mathbb{R}^3) \subseteq \mathcal{B}^{\frac{1}{2}}_{\frac{18}{7},2}(\mathbb{R}^3) \subseteq \mathcal{B}^{\frac{1}{3}}_{3,2}(\mathbb{R}^3).$$

Def. : Weak solution

u is called a weak solution of the incompressible Euler equation w/ initial data $u_0 \in L^2(\mathbb{R}^3)$ iff

- $u \in C_w([0,T];L^2(\mathbb{R}^3))$
- $\forall \varphi \in C^1([0,T];\mathcal{S}(\mathbb{R}^3))$ w/ $\nabla_x \cdot \varphi = 0$ $\forall 0 \leq t \leq T$

$$(u(t),\varphi(t))_{L^2} - (u(0),\varphi(0))_{L^2} - \int_0^t (u(s),\partial_s\varphi(s))_{L^2}\,ds = \int_0^t \int_{\mathbb{R}^3} \left[(u(s)\cdot\nabla)\varphi(s)\right]\cdot u(s)\,d\mu\,ds$$

- $\nabla_x \cdot u(t) = 0$ in the distributional sense for all $0 \leq t \leq T$.

Estimates on energy flux

Define
$$\Pi_Q(t) := \frac{1}{2}\frac{d}{dt}\|S_Q u(t)\|_2^2 \overset{\text{Assume we can interchange } \partial_t \leftrightarrow \int}{=} \int \mathrm{Tr}\left[S_Q(u\otimes u)\cdot \nabla S_Q u\right]dx,$$
$$\left[\mathcal{F}^{-1}\varphi(x) := (2\pi)^{-3}\int_{\mathbb{R}^3} e^{-ikx}\varphi(k)dk. \text{ (FT)}\right]$$

Then we have
$$\Pi_Q(t) = \int \mathrm{Tr}\left[r_Q(u,u)\cdot\nabla S_Q u\right]dx + \int \mathrm{Tr}\left[(u-S_Q u)\otimes(u-S_Q u)\cdot\nabla S_Q u\right]dx$$
w/ $\quad r_Q(u,u) := \int \check{h}_Q(y)\left(u(x-y) - u(x)\right)\otimes\left(u(x-y)-u(x)\right)dy$

and $\quad \check{h}_Q(y) := \lambda_{Q+1}^3\,\check{h}(\lambda_{Q+1}y).$ $\qquad$ Used $\int \check{h}_Q(y)dy = 1$

Note that we used
$$S_Q(u\otimes u) = r_Q(u,u) + (u-S_Q u)\otimes(u - S_Q u) + (S_Q u)^{\otimes 2}$$
together w/
$$\int \mathrm{Tr}\left[S_Q u\otimes S_Q u\cdot\nabla S_Q u\right]dx = \int S_Q u_i\, S_Q u_j\, \partial_j S_Q u_i = -\frac{1}{2}\int |S_Q u|^2 S_Q(\nabla\cdot u) = 0$$
where for $S_Q v\in L^2$ if $\widehat{S_Q v}\in L^2$, we conclude
$$\int S_Q\,\partial_i v_{jk}\, S_Q w_\ell \overset{\text{Plancherel}}{=} \int \widehat{\xi_i S_Q v_{jk}}\,\overline{\widehat{S_Q w_\ell}} = -\int S_Q v_{jk}\, S_Q \partial_i w_\ell .$$

We continue w/
$$|\Pi_Q| \leq \|r_Q(u,u)\|_{3/2}\,\|\nabla S_Q u\|_3 + \|u - S_Q u\|_3^2\,\|\nabla S_Q u\|_3$$
$$\overset{\text{MI}}{\leq} \left[\int \check{h}_Q(y)\,\|u(\cdot-y)-u\|_3^2\,dy + \|u-S_Q u\|_3^2\right]\|\nabla S_Q u\|_3$$

Have
$$\|u(\cdot-y)-u\|_3^2 \overset{\text{LPI}}{\lesssim} \left\|\left(\sum_q |\Delta_q u(\cdot-y)-u|^2\right)^{1/2}\right\|_3^2 \overset{\text{MI}}{\underset{\text{MVT}}{\lesssim}} |y|^2\sum_{q\leq Q}\|\nabla\Delta_q u\|_3^2 + \sum_{q>Q}\|\Delta_q u\|_3^2$$
$$\overset{\text{BI}}{\lesssim} |y|^2\sum_{q\leq Q}\lambda_q^2\|\Delta_q u\|_3^2 + \sum_{q>Q}\|\Delta_q u\|_3^2 .$$

Furthermore, we have

$$\|\nabla S_Q u\|_3^2 \lesssim \Big\|\Big(\sum_k |\Delta_k S_Q \nabla u|^2\Big)^{1/2}\Big\|_3^2 \lesssim \sum_{k\le Q+1} \Big\|\sum_{q=k-1}^{k+1\wedge Q} \nabla \Delta_k \Delta_q u\Big\|_3^2$$

$$\lesssim \sum_{q\le Q} \sum_{q'=q-2}^{(q+2)\wedge Q} \lambda_q \|\Delta_q u\|_3\, \lambda_{q'} \|\Delta_{q'} u\|_3$$

$$\lesssim \sum_{q\le Q} \lambda_q^2 \|\Delta_q u\|_3^2 .$$

In addition,

$$\|u - S_Q u\|_3^2 = \Big\|\sum_{q>Q} \Delta_q u\Big\|_3^2 \lesssim \sum_{q>Q} \|\Delta_q u\|_3^2 .$$

(anal. to above)

Using $\int |y|^{2} h_Q(y)\,dy \approx \lambda_Q^{-2}$, we end up w/

$$|\Pi_Q| \lesssim \Big(\lambda_Q^{-2} \sum_{q\le Q} \lambda_q^2 \|\Delta_q u\|_3^2 + \sum_{q>Q} \|\Delta_q u\|_3^2\Big)\Big(\sum_{q\le Q} \lambda_q^2 \|\Delta_q u\|_3^2\Big)^{1/2}$$

$$= \Big(\lambda_Q^{-2} \sum_{q\le Q} \lambda_q^{4/3} d_q^2 + \sum_{q>Q} \lambda_q^{-\frac{2}{3}} d_q^2\Big)\Big(\sum_{q\le Q} \lambda_q^{4/3} d_q^2\Big)^{1/2}$$

$$= \Big(\lambda_Q^{-\frac{4}{3}} \sum_{q\le Q} \lambda_q^{4/3} d_q^2 + \lambda_Q^{2/3} \sum_{q>Q} \lambda_q^{-\frac{2}{3}} d_q^2\Big)\Big(\sum_{q\le Q} \lambda_q^{4/3} d_q^2 \lambda_Q^{-\frac{4}{3}}\Big)^{1/2},$$

where we introduced $d_q := \lambda_q^{1/3} \|\Delta_q u\|_3$.

In here we used

$$\sum_{q>Q} \lambda_q^{-\frac{2}{3}} d_q^2 \le \sup_{q>Q} d_q^2 \sum_{q>Q} \lambda_q^{-\frac{2}{3}} = \sup_{q>0} d_q^2 \frac{\lambda_{Q+1}^{-\frac{2}{3}}}{1-2^{-2/3}} \lesssim \lambda_Q^{-\frac{2}{3}} \sup_{q>Q} d_q^2 .$$

Lemma 1: Let $b>1$, $(a_q)_{q\geqslant -1}\in c_0$. Then one has

$$b^{-Q}\sum_{q=-1}^{Q} b^q a_q \xrightarrow[Q\to\infty]{} 0.$$

Proof: Let $\varepsilon>0$. Choose $Q_0\in\mathbb{N}$ s.t. $\forall q\geqslant Q_0$ $|a_q|\leqslant\frac{\varepsilon(b-1)}{b}$. Then

$$\left|b^{-Q}\sum_{q=-1}^{Q} b^q a_q\right| \leqslant b^{-Q}\left[\sum_{q=-1}^{Q_0-1} b^q|a_q| + \sum_{q=Q_0}^{Q} b^q|a_q|\right] \leqslant b^{-Q}\left[\underbrace{\sum_{q=-1}^{Q_0-1} b^q|a_q|}_{=:C_{Q_0}} + \frac{\varepsilon(b-1)}{b}\,\frac{b^{Q+1}-b^{Q_0}}{b-1}\right]$$

$$\leqslant C_{Q_0}b^{-Q}+\varepsilon$$

$$\Rightarrow \limsup_{Q\to\infty}\left|b^{-Q}\sum_{q=-1}^{Q} b^q a_q\right|\leqslant\varepsilon.$$

$\blacksquare$

As a consequence, if we assume $d\in c_0$ ($\Leftrightarrow u\in B^{1/3}_{3,c_0(\mathbb{N})}$), we obtain

$$\lim_{Q\to\infty}|\Pi_Q| \lesssim \lim_{Q\to\infty}\left(\lambda_Q^{-\frac{4}{3}}\sum_{q\leqslant Q}\lambda_q^{2/3}d_q^2 + \cancel{\lambda_Q^{2/3}\lambda_Q^{-\frac{2}{3}}}\underbrace{\sup_{q>Q}d_q^2}_{<\infty}\right)\left(\sum_{q\leqslant Q}\lambda_q^{4/3}d_q^2\,\lambda_Q^{-\frac{4}{3}}\right)^{1/2}$$

$$\overset{\text{Lemma}}{=} 0.$$

General comment on sufficient Besov regularity:

Note that we find for a multi-index $\alpha\in\mathbb{N}_0^3$

$$\text{"}\int \mathrm{Tr}\big[\underbrace{S_Q(u\otimes u)\cdot D^\alpha S_Q u}_{\text{not precisely stated}}\big]\text{"} \overset{\text{analog to the above}}{\lesssim}\left(\lambda_Q^{-2}\sum_{q\leqslant Q}\lambda_q^2\|\Delta_q u\|_3^2+\sum_{q>Q}\|\Delta_q u\|_3^2\right)\left(\sum_{q\leqslant Q}\lambda_q^{2|\alpha|}\|\Delta_q u\|_3^2\right)^{\frac{1}{2}}$$

$$=\left(\lambda_Q^{-2+|\alpha|-\beta}\sum_{q\leqslant Q}\lambda_q^{2-2\beta}\eta_q^2+\lambda_Q^{|\alpha|-\beta}\sum_{q>Q}\lambda_q^{-2\beta}\eta_q^2\right)\left(\lambda_Q^{2\beta-2|\alpha|}\sum_{q\leqslant Q}\lambda_q^{2|\alpha|-2\beta}\eta_q^2\right)^{\frac{1}{2}}$$

where $\beta\in\mathbb{R}$ is to be determined and $\eta_q:=\lambda_q^{\beta}\|\Delta_q u\|_3$.
In order to apply the above lemma, we impose

$$\left|\begin{array}{l} 2-|\alpha|+\beta = 2-2\beta \\ -|\alpha|+\beta = -2\beta\end{array}\right| \Leftrightarrow \beta=\frac{|\alpha|}{3}.$$

This justifies the above calculation.

Alternative auxiliary result:

Lemma 2: Let $K \in \ell^1(\mathbb{Z})$, $(a_q)_{q \geq -1} \in \ell^\infty$ be an arbitrary sequence. Then we have

$$\limsup_{Q \to \infty} |K * a(Q)| \leq \|K\|_1 \cdot \limsup_{Q \to \infty} |a(Q)|$$

Proof: Let $\varepsilon > 0$. Fix $M \in \mathbb{N}$ s.t.

$$\left.\begin{aligned} &\sum_{|q| \geq M} |K(q)| < \frac{\varepsilon}{2(\|a\|_\infty + 1)} \\ &\forall q \geq M: \quad |a_q| < \limsup_{k \to \infty} |a_k| + \frac{\varepsilon}{2(\|K\|_1 + 1)} \end{aligned}\right\} (*)$$

Let $Q \geq 2M$. We obtain

$$|K * a(Q)| \leq \sum_{q \leq M} \underbrace{|K(Q-q)|}_{\geq M} \underbrace{|a_q|}_{\leq \|a\|_\infty} + \sum_{q > M} |K(Q-q)| |a_q|$$

$$\overset{(*)}{\leq} \|K\|_1 \cdot \limsup_{k \to \infty} |a_k| + \varepsilon .$$

$\varepsilon \downarrow 0$ yields the result.

$\blacksquare$

In particular, if we define

$$K(q) := \begin{cases} \lambda_q^{-4/3}, & q \geq 0 \\ \lambda_q^{2/3}, & q < 0 \end{cases},$$

we obtain by the above estimates

$$|\Pi_Q| \lesssim \left(K * d_q^2(Q)\right)^{3/2}$$

w/ $K \in \ell^1(\mathbb{Z})$. Thus the above lemma is applicable and yields

$$\limsup_{Q \to \infty} |\Pi_Q| \lesssim \limsup_{Q \to \infty} d_q^3 = 0.$$

Energy flux between shells

Define

$$\Pi_{Q_0Q_1} := \frac{1}{2}\frac{d}{dt}\|S_{Q_0Q_1}u\|_2^2 = \int \mathrm{Tr}\left[S_{Q_0Q_1}(u\otimes u)\cdot\nabla S_{Q_0Q_1}u\right]dx$$

w/ $S_{Q_0Q_1} := S_{Q_1} - S_{Q_0-1}$.

Formal computation:

$$S_{Q_0Q_1}^2 = (S_{Q_1} - S_{Q_0-1})^2 = S_{Q_1}^2 + S_{Q_0-1}^2 - 2\underbrace{S_{Q_0-1}S_{Q_1}}_{\text{projections commute}}$$

since $\Delta_q\Delta_{q'} = 0$ if $|q-q'| \geq 2$ (*)

$$= S_{Q_1}^2 - S_{Q_0-1}^2 - 2S_{Q_0-1}(S_{Q_0} - S_{Q_0-1}) = S_{Q_1}^2 - S_{Q_0-1}^2 - 2\Delta_{Q_0-1}\Delta_{Q_0}$$

This leads to

$$\Pi_{Q_0Q_1} = \Pi_{Q_1} - \Pi_{Q_0-1} - 2\int_{\mathbb{R}^3}\mathrm{Tr}\left[\bar\Delta_{Q_0}(u\otimes u)\cdot\nabla\bar\Delta_{Q_0}u\right]dx$$

w/

$$\bar\Delta_{Q_0}v(x) := \int \bar h_{Q_0}(y)v(x-y)\,dy, \qquad h_{Q_0} := \mathcal{F}^{-1}\sqrt{\varphi(\lambda_{Q_0-1}^{-1}\cdot)\varphi(\lambda_{Q_0}^{-1}\cdot)}.$$

As before, we decompose the last term on which we do not already have an upper bound using

$$\bar\Delta_{Q_0}(u\otimes u)(x) = \underbrace{\int \bar h_{Q}(y)\big(u(x-y)-u(x)\big)\otimes\big(u(x-y)-u(x)\big)dy}_{=:\bar r_{Q_0}(u,u)} + \bar\Delta_{Q_0}u(x)\otimes u(x) + u(x)\otimes\bar\Delta_{Q_0}u(x) - u(x)\otimes u(x)$$

This leads to

$$\int\mathrm{Tr}\left[\Delta_{Q_0}(u\otimes u)\cdot\nabla\bar\Delta_{Q_0}u\right]dx = \underbrace{\int\mathrm{Tr}\left[\bar r_{Q_0}(u,u)\cdot\nabla\bar\Delta_{Q_0}u\right]dx}_{=:I} + \underbrace{\int\mathrm{Tr}\left[\bar\Delta_{Q_0}u(x)\otimes u(x)\cdot\nabla\bar\Delta_{Q_0}u\right]dx + \int\mathrm{Tr}\left[u\otimes\bar\Delta_{Q_0}u\cdot\nabla\bar\Delta_{Q_0}u\right]dx}_{=:II}$$

S_{Q_0} (see below)

$$\underbrace{-\int\mathrm{Tr}\left[\overset{\sum_q \Delta_q u}{\overset{\parallel}{u}}\otimes u\cdot\nabla\bar\Delta_{Q_0}u\right]dx}_{=:III}$$

$q \geq Q_0+1$ drops

Have $|I| \leq \|\bar r_{Q_0}(u,u)\|_{3/2}\|\nabla\bar\Delta_{Q_0}u\|_3$

where as above

$$\|\bar r_{Q_0}(u,u)\|_{3/2} \leq \int|\bar h_{Q_0}(y)|\,\|u(\cdot-y)-u\|_3^2 \lesssim \underbrace{\int|\bar h_{Q_0}(y)||y|^2dy}_{\lesssim\lambda_{Q_0}^{-2}}\sum_{q\leq Q_0}\lambda_q^2\|\Delta_q u\|_3^2 + \sum_{q>Q_0}\|\Delta_q u\|_3^2$$

$$\|\nabla\bar\Delta_{Q_0}u\|_3^2 \lesssim \sum_{k=Q_0-1}^{Q_0}\underbrace{\|\Delta_k\nabla\bar\Delta_{Q_0}u\|_3^2}_{\lesssim\|\nabla\Delta_k u\|_3^2} \lesssim \lambda_{Q_0-1}^2\|\Delta_{Q_0-1}u\|_3^2 + \lambda_{Q_0}^2\|\Delta_{Q_0}u\|_3^2$$

$$\Rightarrow |\mathrm{I}| \lesssim \Big(\underbrace{\lambda_{Q_0}^{-\frac{4}{3}} \sum_{q\le Q_0} \lambda_q^{\frac{4}{3}} d_q^2}_{\lesssim \lambda_{Q_0}^{-\frac{2}{3}} \sum_{q\le Q_0} \lambda_q^{\frac{2}{3}} d_q^2} + \lambda_{Q_0}^{\frac{2}{3}} \sum_{q>Q_0} \lambda_q^{-\frac{2}{3}} d_q^2 \Big)\Big(d_{Q_0-1}^2 + d_{Q_0}^2 \Big)^{\frac{1}{2}}.$$

due to $\lambda_{\underset{\ge 0}{Q_0-q}}^{-\frac{4}{3}} \le \lambda_{Q_0-q}^{-2/3}$ done.

$$|\mathrm{II}| \lesssim \| \bar{\Delta}_{Q_0} u \otimes S_{Q_0} u \|_{3/2} \, \| \nabla \bar{\Delta}_{Q_0} u \|_3 \le \| \bar{\Delta}_{Q_0} u \|_3 \, \| S_{Q_0} u \|_3 \, \| \nabla \bar{\Delta}_{Q_0} u \|_3 \lesssim \Big(d_{Q_0-1}^2 + d_{Q_0}^2 \Big) \Big(\lambda_{Q_0}^{-\frac{2}{3}} \sum_{q\le Q_0} \lambda_q^{\frac{2}{3}} d_q^2 \Big)^{\frac{1}{2}}$$

$$|\mathrm{III}| \lesssim \| S_{Q_0} u \|_3^2 \, \| \nabla \bar{\Delta}_{Q_0} u \|_3 \lesssim \Big(\lambda_{Q_0}^{-\frac{2}{3}} \sum_{q\le Q_0} \lambda_q^{\frac{2}{3}} d_q^2 \Big) \Big(d_{Q_0-1}^2 + d_{Q_0}^2 \Big)^{\frac{1}{2}}.$$

Using $\lambda_{Q-q}^{-\frac{4}{3}} \le \lambda_{Q-q}^{-2/3}$ in the above estimates on $|\Pi_Q|$ together w/ $\bar{K}(q) := \lambda_{|q|}^{-\frac{2}{3}}$, we arrive at

$$|\Pi_{Q_0 Q_1}| \lesssim \big(\bar{K} * d^2 \big)^{3/2}(Q_0) + \big(\bar{K} * d^2 \big)^{3/2}(Q_1),$$

where we also defined $d^2 := (d_q^2)_{q\ge -1}$.

Estimate on helicity

The helicity is def. as

$$\mathcal{H} := \int u \cdot \underbrace{\nabla \times u}_{\omega} \, dx$$

and the LP helicity $\mathcal{H}_Q$ satisfies

$$h_Q := \frac{d}{dt} \mathcal{H}_Q = \frac{d}{dt} \int S_Q u \cdot S_Q \omega \, dx = \varepsilon_{ijk} \int \partial_t \left(S_Q u_i \, S_Q \partial_j u_k \right) dx$$

$$= \varepsilon_{ijk} \int \left[S_Q(\partial_t u_i) S_Q(\partial_j u_k) + \overset{-\partial_j}{S_Q u_i} \, S_Q(\partial_j \partial_t u_k) \right]$$

$i \leftrightarrow k$ yields add. (-1) due to

$$= 2 \varepsilon_{ijk} \int S_Q(\partial_t u_i) S_Q(\partial_j u_k) = -2 \varepsilon_{ijk} \int S_Q\left(u_\ell \partial_\ell u_i\right) S_Q(\partial_j u_k) \, dx$$

$$= 2 \int \mathrm{Tr}\left[S_Q(u \otimes u) \cdot \nabla S_Q \omega \right].$$

Using the above estimates on $S_Q(u \otimes u)$ from the energy estimate, we find

$$|h_Q| \lesssim \left(\cancel{\|S_Q(u\otimes u)\|_{3/2} + \|u - S_Q u\|_3} \right) \|\nabla S_Q \omega\|_3 \lesssim \left(\lambda_Q^{-2} \sum_{q \le Q} \lambda_q^2 \|\Delta_q u\|_3^2 + \sum_{q > Q} \|\Delta_q u\|_3^2 \right) \left(\sum_{q \le Q} \|\nabla \Delta_q \omega\|_3^2 \right)^{\frac{1}{2}}.$$

Using the two estimates above, we obtain

$$|h_Q| \lesssim \left(\lambda_Q^{-\frac{2}{3}} \sum_{q \le Q} \lambda_q^{\frac{2}{3}} b_q^2 + \lambda_Q^{\frac{4}{3}} \sum_{q > Q} \lambda_q^{-\frac{4}{3}} b_q^2 \right) \left(\lambda_Q^{-\frac{8}{3}} \sum_{q \le Q} \lambda_q^{8/3} b_q^2 \right)^{1/2}$$

$$\lesssim \left(K_b * b^2(Q) \right)^{3/2}$$

where we defined $b^2 := (b_q^2)_{q \ge -1}$, $K_b(q) := \begin{cases} \lambda_q^{-\frac{2}{3}} & q \ge 0 \\ \lambda_q^{4/3} & q < 0 \end{cases}$.

We have $K_b \in \ell^1(\mathbb{Z})$ and thus Lemma 2 yields

$$\limsup_{Q \to \infty} |h_Q| \lesssim \limsup_{Q \to \infty} b_Q^3 = 0$$

provided $(b_q)_{q \ge -1} \in c_0$, i.e., $u \in B^{\frac{2}{3}}_{3, c_0}$.

Estimate on enstrophy

Define

$$P_\xi^\perp := I_3 - |\xi|^{-2}\,\xi \otimes \xi \quad , \ \xi \in \mathbb{R}^3$$

and

$$\nabla^\perp u := \mathcal{F}^{-1}(P_\cdot^\perp \mathcal{F}u),$$

and also the enstrophy

$$\Omega_Q := \frac{1}{2}\frac{d}{dt}\|S_Q\omega\|_2^2 = \int \mathrm{Tr}\left[S_Q(u\otimes u)\cdot \nabla\nabla^\perp S_Q\omega\right]dx.$$

Employing the above estimates, we immediately obtain

$$|\Omega_Q| \lesssim \left(\|r_Q(u,u)\|_{3/2} + \|u - S_Q u\|_3^2\right)\|\nabla\nabla^\perp S_Q\omega\|_3$$

$$\lesssim \left(\lambda_Q^{-2}\sum_{q\le Q}\|\Delta_q\omega\|_3^2 + \sum_{q>Q}\|\Delta_q u\|_3^2\right)\left(\sum_{q\le Q}\lambda_q^4\|\Delta_q\omega\|_3^2\right)^{\frac{1}{2}}$$

more terms ≥ 0 $\quad \sum \lambda_q^{-2}\|\Delta_q\omega\|_3^2$

$$\leq \left(\lambda_Q^{-2}\|S_Q\omega\|_3^2 + \sum_{q>Q}\lambda_q^{-2}c_q^2\right)\left(\sum_{q\le Q}\lambda_q^4 c_q^2\right)^{\frac{1}{2}},$$

where we defined $c_q := \|\Delta_q\omega\|_3$. If we further define

$$\omega(q) := \begin{cases}\lambda_q^{-4} & , q \geq 0\\ \lambda_q^2 & , q < 0\end{cases},$$

we arrive at

$$|\Omega_Q| \lesssim \|S_Q\omega\|_3^2\,(\omega * \underline{c}^2)^{1/2}(Q) + (\omega * c^2)^{3/2}(Q).$$

analogously def. to above

Lower bound on used estimate

ℓ^∞ sufficient

Lemma: Let $(a_q)_{q\geq -1} \in \ell^1$, $\begin{Bmatrix} f \\ g \end{Bmatrix} : \mathbb{R} \to \mathbb{R}^+$ be some strict exponential $\begin{Bmatrix} \text{decay} \\ \text{growth} \end{Bmatrix}$. Then we have for any $N \in \mathbb{Z}^{\geq -1}$ large enough, $p \in [1, \infty]$, $\alpha \in [0,1]$

(i) $$\inf_{q>N} \left(f(q)^{1-\alpha} a_q \right) f(N)^\alpha \lesssim \sum_{q>N} f(q) a_q \lesssim f(N) \, \|(a_q)_{q\geq N}\|_p$$

(ii) $$\max_{q\leq N} a_q g(q) \lesssim \sum_{q \leq N} g(q) a_q \lesssim g(N) \, \|(a_q)_{q \geq -1}\|_p$$

Proof: First of all, we prove the upper bounds. If $p = \infty$ or $p' = \infty$, both estimates are almost obvious. For $p, p' < \infty$ we obtain

$$\sum_{q>N} f(q) a_q \overset{\text{Hölder}}{\lesssim} \|(a_q)_{q>N}\|_p \left(\sum_{q>N} f(qp') \right)^{\frac{1}{p'}} \overset{\text{geom. series}}{\lesssim} \|(a_q)_{q>N}\|_p \left(\frac{f((N+1)p')}{1 - f(p')} \right)^{\frac{1}{p'}}$$

$$\lesssim \|(a_q)_{q>N}\|_p \, f(N)$$

as well as

$$\sum_{q \leq N} g(q) a_q \leq \|(a_q)_{q\geq -1}\|_p \left(\sum_{q\leq N} g(qp') \right)^{\frac{1}{p'}} = \|(a_q)_{q\geq -1}\|_p \left(\frac{g((N+1)p') - 1}{g(p') - 1} + g(-p') \right)^{\frac{1}{p'}}$$

$$\lesssim \|(a_q)_{q\geq -1}\|_p \, g(N).$$

As the lower bound of (ii) is also obvious, we will only show the lower bound of (i):

$$\sum_{q>N} f(q) a_q \geq \inf_{q>N} \left(f(q)^{1-\alpha} a_q \right) \sum_{q>N} f(q)^\alpha = \inf_{q>N} \left(f(q)^{1-\alpha} a_q \right) \frac{f((N+1)\alpha)}{1 - f(\alpha)}$$

$$\gtrsim \inf_{q>N} \left(f(q)^{1-\alpha} a_q \right) f(N)^\alpha.$$

$\square$