Some useful functional analytic tools in mathematical Physics

<u>Motivation</u>: Is the Hamiltonian describing an electron travelling through a periodic lattice s.a.? Is the system stable?

1. IMS Localization

<u>Reference</u>: H.L. Cycon, B. Simon et al.: "Schrödinger operators w/ Application to QM and Global Geometry."

<u>Def.</u>: $(J_a)_{a\in\mathcal{A}} \in \mathcal{C}^\infty(\mathbb{R}^d)^{\mathcal{A}}$ is a <u>partition of unity</u> iff

(i) $0 \leq J_a \leq 1$ on $\mathbb{R}^d$

(ii) $\{J_a\}_{a\in\mathcal{A}}$ is locally finite, i.e., $\forall K$ cpt $J_a = 0$ <u>a.a.</u> $a \in \mathcal{A}$
all but finitely many

(iii) $\sum_a J_a^2 = 1$

(iv) $\sup \sum_{a\in\mathcal{A}} |\nabla J_a|^2 < \infty$

<u>Thm</u> (IMS Localization formula)

Let $H_0 := -\Delta$, $(J_i)_{i\in\mathcal{I}}$ be a partition of unity. Then

$$H_0 = \sum_{i\in\mathcal{I}} J_i H_0 J_i - \sum_{i\in\mathcal{I}} |\nabla J_i|^2.$$

Proof: Have

$$\left[J_i,[J_i,H_0]\right] = -2|\nabla J_i|^2$$
$$\|$$
$$J_i^2 H_0 + H_0 J_i^2 - 2 J_i H_0 J_i$$

$\sum_{i\in I}$, done (use loc. fin.)

$\boxtimes$

Corollary: Let $V \in L^1_{loc}(\mathbb{R}^d)$, and assume for a partition of unity $(J_i)_i$ that there are $a,b \in \mathbb{R}$ s.t.

$$|V|J_i^2 \leq a\, J_i H_0 J_i + b J_i^2 \quad \forall i. \quad \text{(in the sense of quadratic forms on } Q(H_0)=H^1\text{, see below)}$$

Then

$$|V| \leq a H_0 + \tilde{b}$$

w/ $\tilde{b} := b + \sup \sum |\nabla J_i|^2$.

Proof: $$|V| = \sum_i |V| J_i^2 \leq a \sum_i J_i H_0 J_i + b \overset{\text{IMS}}{=} a H_0 + \tilde{b}$$

$\boxtimes$

Def.: (Ruelle-Simon partition of unity)

$(J_a)_a$ is a RS partition of unity iff

(i) it is indexed by all two-cluster decompositions ($\#a = 2$)

$a = \{A_1, \dots, A_k\}$, $\bigcup_{i=1}^k A_i = \{1,\dots,N\}$, $A_i \cap A_j = \emptyset$ if $i \neq j$ "k-cluster"

(ii) $J_a(\lambda x) = J_a(x)$ $\forall \lambda > 1$, $|x| = 1$

(iii) $\exists C > 0$: $\operatorname{supp}(J_a) \cap \overline{B_1}^c \subseteq \{ x \in \mathbb{R}^{dN} \mid |x_i - x_j| > C|x| \ \forall (i,j) \notin a \}$

$x = (x_1, \dots, x_N)$, $x_N \in \mathbb{R}^d$

pairs not belonging to same cluster

Some preliminary notation: Consider N particles in $\mathbb{R}^d$ w/ masses $\{m_i\}$. The free Hamiltonian is given by

$$\tilde{H}_0 = -\sum_{i=1}^{N} \frac{1}{2m_i} \Delta_{x_i}.$$

Define $X := \{x \in \mathbb{R}^{Nd} \mid \sum_i m_i x_i = 0\} \cong \mathbb{R}^{(N-1)d}$ and introduce relative coordinates $y_i := x_i - x_N$, $1 \le i \le N-1$, and the center of mass $R(x) := \frac{1}{M} \sum_i m_i x_i$, $M := \sum_i m_i$. Then write

$$\tilde{H}_0 =: \left(-\frac{1}{2M} \Delta_R\right) \otimes Id_{L^2(X)} + Id_{L^2(\mathbb{R}^d)} \otimes H_0$$

and define $H := H_0 + V$ w/ $V(x) := \sum_{i<j} V_{ij}(x)$.

Next, def. for a two-cluster decomposition a

- $I_a := \sum_{(ij) \subseteq a} V_{ij}(x)$, $H(a) := H - I_a$
- $\Sigma(a) := \inf \sigma(H(a))$
- $\Sigma := \min_{\# a = 2} \Sigma(a)$.

Thm (HVZ)

With the above notation, have

$$\sigma_{ess}(H) = [\Sigma, \infty)$$

Proof: "$\supseteq$": Easy, using a (singular) Weyl sequence. [RS I, VII.12, p. 292]

"$\subseteq$":
$$H \overset{IMS}{=} \sum_a J_a \underbrace{H(a) J_a}_{\text{RSPU}} + \sum_a I_a J_a^2 - \sum_a |\nabla J_a|^2 .$$

Can show: $I_a J_a$, $|\nabla J_a|$ both rel. cpt w.r.t. H_0 (i.e., $A(H_0+1)^{-1}$ cpt)

Weyl's Thm [RS IV, XIII.14, p. 295] thus gives

rel. cpt perturbations don't change the spectra

$$\sigma_{ess}(H) = \sigma_{ess}\Big(\sum_a J_a \underbrace{H(a)}_{\geq \Sigma(a) \geq \Sigma} J_a\Big) \subseteq [\Sigma, \infty).$$

☒

2. KLMN Thm

Ref. [RS II], [BS], both Chapter X

Recall: Kato-Rellich Thm: [RS II]

Let A be s.a., B be symmetric on $\mathcal{D}(A)$ $\subseteq \mathcal{D}(B)$ and
$$\|B\varphi\| \leq a\|A\varphi\| + b\|\varphi\| \quad \forall \varphi \in \mathcal{D}(A) \quad (*)$$
for some $0<a<1$, $b \in \mathbb{R}$. Then $(A+B, \mathcal{D}(A))$ is s.a. and essentially s.a. on any core. If A is bdd from below, then so is $A+B$ with bound explicitly given, dep. on A, a, b.

What to do in case there is no explicit operator B, but only a quadratic form associated to an interaction?

E.g.: Electron-phonon coupling: "$\int a^*(k) e^{ikx} dk + c.c.$"

Or more generally, if we don't have $(*)$, but something weaker:

$\exists\, 0<a<1, b\in\mathbb{R}$ s.t. for any $\varphi\in\mathcal{D}(A)$

$$\langle\varphi, B\varphi\rangle \leq a\langle\varphi, A\varphi\rangle + b\|\varphi\|^2$$

"rel. form bdd"

Write $\quad B\leq aA+b$ in the sense of quadr. forms.

<u>Def.</u> (Associated form)

Define for a s.a. operator A bdd from below by $-b$ $(A\geq -b)$

$$\langle\varphi,\psi\rangle_{q_A} := \langle\varphi, A\psi\rangle + (1+b)\langle\varphi,\psi\rangle \quad \forall\varphi,\psi\in\mathcal{D}(A),$$

$$Q(A) := \overline{\mathcal{D}(A)}^{\|\cdot\|_{q_A}} \text{ (completion)}.$$

We call the (unique) cont. extension q_A of $\mathcal{D}(A)^2\to\mathbb{C}$, $(\varphi,\psi)\mapsto\langle\varphi, A\psi\rangle$ to $Q(A)$ the (<u>quadr.</u>) <u>form associated</u> w/ A.

<u>Ex.:</u> $A=-\Delta$

$\underset{C_c^\infty}{\mathcal{D}(A)} = H^2$, $Q(A)=H^1$, $\|\cdot\|_{q_A}\simeq\|\cdot\|_{H^1}$.

<u>Rem.:</u> s.a. pos. operator $\overset{1\text{-to-}1}{\longleftrightarrow}$ pos., closed quadr. form

<u>Thm</u> (KLMN) [RS II, X.17, p.167] ↳ w.r.t. induced topology

Let $A\geq 0$ be s.a., β be a symm. quadr. form on $Q(A)$ s.t.

$$|\beta(\varphi,\varphi)| \leq a\langle\varphi, A\varphi\rangle + b\|\varphi\|^2 \quad \forall\varphi\in\mathcal{D}(A)$$

some $0<a<1$, $b\in\mathbb{R}$.

<u>Then</u> $\exists!\, C$ s.a. w/
- $\mathcal{D}(C)=\mathcal{D}(A)$
- $Q(C)=Q(A)$
- $\langle\varphi, C\psi\rangle = \langle\varphi, A\psi\rangle + \beta(\varphi,\psi)\ \forall\varphi,\psi\in\mathcal{D}(A)$

$C\geq -b$ and any domain of ess. s.a. of A is a form core for C

dense subset $\subseteq Q(C)$

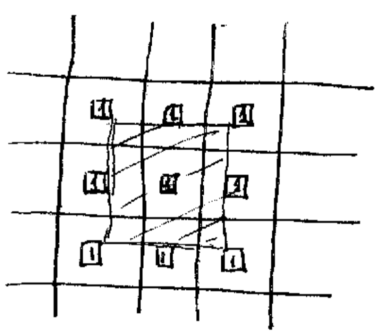
Prop.: If $V \in L^{\frac{3}{2}+\varepsilon}_{\mathrm{loc,un}}(\mathbb{R}^3)$ $\left(:\Leftrightarrow \sup_{\substack{Q \subseteq \mathbb{R}^3 \\ \text{cube of} \\ \text{side length } 1}} \int_Q |V|^{\frac{3}{2}+\varepsilon} dx < \infty\right)$, then

$$H := -\Delta + V \text{ is s.a. on } H^2(\mathbb{R}^3) \text{ and bdd from below.}$$

Proof: Let $(Q_i)_i$ be a sequence of closed unit cubes s.t. $Q_i^\circ \cap Q_j^\circ = \emptyset$, $i \neq j$, and

$$\mathbb{R}^3 = \bigcup_i Q_i$$

and $(J_i)_i$ be a part. of unity s.t. $J_i\big|_{\substack{\text{cube side length } \frac{1}{2}, \\ \text{center} = \text{center}(Q_i)}} = 1$; $J_i\big|_{\underbrace{\left(\substack{\text{cube side} \\ \text{length } \frac{3}{2} \\ \text{center} = \text{center}(Q_i)}\right)}_{=: \tilde{Q}_i}{}^{c}} = 0$



Get

$$|\langle \varphi, V J_i^2 \varphi \rangle| \leq \int_{\tilde{Q}_i} |V| |J_i \varphi|^2 dx \overset{\text{Hölder}}{\leq} \underbrace{\|V\|_{L^{\frac{3}{2}+\varepsilon}(\tilde{Q}_i)}}_{\leq 3^3 \|V\|_{L^{\frac{3}{2}+\varepsilon}_{\mathrm{loc,un}}}} \|J_i \varphi\|^2_{\frac{(3+2\varepsilon)2}{1+2\varepsilon}}$$

$$\overset{\text{GNS}}{\leq} C \|V\|_{L^{\frac{3}{2}+\varepsilon}_{\mathrm{loc,un}}} \|\nabla (J_i \varphi)\|_2^{\frac{6}{3+2\varepsilon}} \|J_i \varphi\|^{\frac{4\varepsilon}{3+2\varepsilon}}$$

$$\left.\begin{array}{l} p = \frac{3+2\varepsilon}{3} \\ q = \frac{3+2\varepsilon}{2\varepsilon} \end{array}\right\} \frac{q}{p} = \frac{3}{2\varepsilon}$$

$$\leq \frac{\lambda}{p} \|\nabla (J_i \varphi)\|_2^2 + \frac{\left(C \|V\|_{L^{\frac{3}{2}+\varepsilon}_{\mathrm{loc,un}}}\right)^q}{\lambda^{q/p} q} \|J_i \varphi\|_2^2 .$$

Corollary of IMS Loc. + KLMN $\Longrightarrow$ ∎