Geometric convergence

Ref.: H Lewin: "Geometric methods for nonlinear many-body quantum systems", JFA 260 (2011), pp. 3535 - 3595.

Motivation: Have minimization problem $E = \inf_{\|\Psi\|=1} \langle \Psi, H\Psi \rangle$.

⇝ Want to use min. sequence (Ψ_n) to obtain useful estimates on E.

Problem: Only have weak-convergent subsequence $(\Psi_{n_k})_k$ and limit could even be 0.

Solution: Introduce a weaker concept of convergence.

⇝ Need appropriate language.

Second quantization

1) Underlying spaces

For (sep.) H-space $\mathcal{H} = L^2(\mathbb{R}^d)$, def. Schatten space $\mathfrak{S}_p(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$ by req.

$$\|A\|_{\mathfrak{S}_p(\mathcal{H})} := \mathrm{Tr}_{\mathcal{H}}(|A|^p)^{\frac{1}{p}} < \infty$$

$p=1$: Trace class, $p=2$: HS. Have

$$(\underbrace{\mathcal{K}(\mathcal{H})}_{c_0})' = \underbrace{\mathfrak{S}_1(\mathcal{H})}_{\ell^1}, \quad \mathfrak{S}_1(\mathcal{H})' = \underbrace{\mathcal{L}(\mathcal{H})}_{\ell^\infty}$$

$\mathcal{H}$ sep. → $\mathcal{K}(\mathcal{H})$ sep. ⟹ $(\Gamma_n)_n \in \left(B_1^{\mathfrak{S}_1(\mathcal{H})}\right)^{\mathbb{N}}$ has weak-* conv. subseq. i.e.,

$$\mathrm{Tr}((\Gamma_n - \Gamma)K) \xrightarrow[n\to\infty]{} 0 \quad \forall K \in \mathcal{K}(\mathcal{H}).$$

Define many-particle spaces

$$\mathcal{H}^N_{a/s} := \begin{Bmatrix} \bigwedge \\ \bigvee \end{Bmatrix}_1^N \mathcal{H} =: \mathcal{H}^{\otimes_{a/s} N} .$$

For the particle number may not be conserved (later), we introduce the graded algebra

$$\mathcal{F}_{a/s} := \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathcal{H}^n_{a/s} \text{ , called } \underline{\text{Fock space}}.$$

→ becomes H-space when endowed w/

$$\langle \Psi_1, \Psi_2 \rangle_{\mathcal{F}_{a/s}} := \sum_{n \geq 0} \langle \Psi_1^{(n)}, \Psi_2^{(n)} \rangle_{\mathcal{H}^n_{a/s}} .$$

$\mathcal{H}^0_{a/s} = \mathbb{C}$

Def. $\underline{\text{vacuum}}$ $\Omega := (1, 0, \ldots, 0) \in \mathcal{F}_{a/s}$. Introduce the $\underline{\text{truncated Fock space}}$:

$$\mathcal{F}^{\leq N}_{a/s} := \mathbb{C} \oplus \bigoplus_{n=1}^{N} \mathcal{H}^n_{a/s} .$$

$\mathcal{F}_{a/s}$ becomes graded algebra if one defines for $\varphi_i \in \mathcal{H}^{N_i}_{a/s}$

$$\varphi_1 \begin{Bmatrix} \wedge \\ \vee \end{Bmatrix} \varphi_2 (x_1, \ldots, x_{N_1+N_2}) := \frac{1}{\sqrt{N_1! N_2! (N_1+N_2)!}} \sum_{\pi \in S_{N_1+N_2}} \begin{Bmatrix} \mathrm{sign}(\pi) \\ 1 \end{Bmatrix} \varphi_1 \otimes \varphi_2 (x_{\pi(1)}, \ldots, x_{\pi(N_1+N_2)})$$

where we refer to $\begin{Bmatrix} a \\ s \end{Bmatrix}$ in the $\begin{Bmatrix} \text{fermionic} \\ \text{bosonic} \end{Bmatrix}$ case.

Note that if $\{f_i\} \subseteq \mathcal{H}$ is an ONB, we have that
$$\begin{cases} \{f_{i_1} \wedge \ldots \wedge f_{i_N}\}_{i_1 < \ldots < i_N} \text{ is an ONB of } \mathcal{H}^N_a \\ \{f_{i_1} \vee \ldots \vee f_{i_N}\}_{i_1 \leq \ldots \leq i_N} \text{ is an orthogonal basis of } \mathcal{H}^N_s \end{cases}$$
(Have $f^{\vee N} = \sqrt{N!}\, f^{\otimes N}$)

2) Annihilation and creation operators

Let $f \in \mathcal{H}$, $\mathcal{F}_{a/s}^{fin} := \bigcup_{N \geq 1} \mathcal{F}_{a/s}^{\leq N}$.

Def. $a^+(f) : \mathcal{F}_{a/s}^{fin} \to \mathcal{F}_{a/s}^{fin}$ by requiring:

$$(i)\ a^+(f) \underbrace{\mathcal{H}_{a/s}^N}_{} \subseteq \mathcal{H}_{a/s}^{N+1} \quad \forall N \geq 0$$

abuse of notation: $0 \oplus .. \oplus 0 \oplus \mathcal{H}_{a/s}^N \oplus 0 .. \subseteq \mathcal{F}_{a/s}$

$$(ii)\ \forall \psi \in \mathcal{H}_{a/s}^N \quad a^+(f)\psi := \begin{cases} f \wedge \psi & \text{fermions} \\ f \vee \psi & \text{bosons} \end{cases}$$

If $\{f_i\}_i$ is an ONB of $\mathcal{H}$, then $\begin{Bmatrix} \{\prod_{k=1}^N a^+(f_{i_k})\Omega\}_{i_1 < .. < i_N, N \geq 0} \\ \{ - \,\text{"}\, - \}_{i_1 \leq .. \leq i_N, N \geq 0} \end{Bmatrix}$ is an $\begin{Bmatrix} \text{ON} \\ \text{orthogonal} \end{Bmatrix}$ basis of $\mathcal{F}_{a/s}$.

Def. $a(f) : \mathcal{F}_{a/s}^{fin} \to \mathcal{F}_{a/s}^{fin}$ by requiring:

$$(i)\ a(f) \mathcal{H}_{a/s}^N \subseteq \mathcal{H}_{a/s}^{N-1} \quad \forall N \geq 1 \ \underline{\text{and}} \ a(f)\Omega = 0$$

$$(ii)\ \forall \psi \in \mathcal{H}_{a/s}^N \quad (a(f)\psi)(x_1, .., x_{N-1}) := \sqrt{N} \int \overline{f(x)}\, \psi(x, x_1, .., x_{N-1})\, dx.$$

$a^+(f)$ is called <u>creation</u> operator, $a(f)$ <u>annihilation</u> operator. We abuse notation in the sense of

$$\left(a^+(f)\left((\psi^{(n)})_{n \in \mathbb{N}_0}\right)\right)^{(j+1)} := a^+(f)\psi^{(j)},$$

$$\left(a(f)\left((\psi^{(n)})_{n \in \mathbb{N}_0}\right)\right)^{(j)} := a(f)\psi^{(j+1)}.$$

We have
$$\langle \Phi, a^+(f)\Psi \rangle_{\mathcal{F}_{a/s}} = \langle a(f)\Phi, \Psi \rangle_{\mathcal{F}_{a/s}} \rightsquigarrow \text{justifies notation.}$$

In the fermionic case have (CAR): $\begin{cases} \{a(g), a^+(f)\} = \langle g, f \rangle \mathbb{1}_{\mathcal{F}_a} \\ \{a^+(f), a^+(g)\} = \{a(f), a(g)\} = 0 \end{cases}$ $\quad (\{A,B\} := AB + BA)$

In the bosonic case have (CCR): Exchange $\{\cdot,\cdot\}$ by $[\cdot,\cdot]$ where $[A,B] := AB - BA$.

3) Observables

Start w/ occupation number operator

$$\mathcal{N} := \bigoplus_{N\geq 0} N, \quad \mathcal{D}(\mathcal{N}) = \left\{ (\psi^{(N)})_{N\geq 0} \in \mathcal{F}_{a/s} \;\middle|\; \sum_{N\geq 0} N^2 \|\psi^{(N)}\|^2_{\mathcal{H}^N_{a/s}} < \infty \right\}$$

For $A : \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{H}$ s.a. def.

$$\hat{A} := \mathbb{A} := d\Gamma(A) := 0 \oplus \bigoplus_{N\geq 1} \left(\sum_{i=1}^{N} A_{x_i} \right)$$

$\mathbb{A}$ is s.a. if

$$\mathcal{D}(\mathbb{A}) := \left\{ (\psi^{(N)})_{N\geq 0} \in \bigoplus_{N\geq 0} \underbrace{\mathcal{D}\Big(\sum_{j=1}^{N} A_{x_j}\Big)}_{\substack{A \text{ bdd} \\ = \\ \text{for bdd}}\left(\bigwedge_{\bigvee}\right)_i^N \mathcal{D}(A)} \;\middle|\; \sum_{N\geq 0} \Big\| \Big(\sum_{j=1}^{N} A_{x_j}\Big) \psi^{(N)} \Big\|^2_{\mathcal{H}^N_{a/s}} < \infty \right\}.$$

Note that $\mathcal{N} = d\Gamma(\mathrm{Id}_{\mathcal{H}})$. Take $\{f_i\}_{i\geq 1} \subseteq \mathcal{D}(A)$ to be an ONB of $\mathcal{H}$. Then we have

$$\mathbb{A} = \sum_{j\geq 1} a^*(Af_j)\, a(f_j) = \sum_{i,j\geq 1} \underbrace{A_{ij}}_{= \langle f_i, Af_j \rangle_{\mathcal{H}}} a^*(f_i)\, a(f_j)$$

For a two-body operator $W : \mathcal{H}^2_{a/s} \to \mathcal{H}^2_{a/s}$ def.

$$\mathbb{W} := 0 \oplus 0 \oplus \bigoplus_{N\geq 2} \Big(\sum_{1\leq i<j\leq N} W_{ij} \Big)$$

W_{ij} acting on x_i and x_j.

Again w/ ONB $\{f_i\}$

$$\mathbb{W} = \sum_{\substack{1\leq k\leq l \\ 1\leq i\leq j}} W_{ijkl}\, a^*(f_i)\, a^*(f_j)\, a(f_l)\, a(f_k)$$

w/
$$W_{ijkl} := \begin{cases} \langle f_i \wedge f_j, W f_k \wedge f_l \rangle_{\mathcal{H}^2_a} & \text{fermions} \\[2ex] \dfrac{\langle f_i \vee f_j, W f_k \vee f_l \rangle_{\mathcal{H}^2_s}}{(1+\delta_{ij})(1+\delta_{kl})} & \text{bosons} \end{cases}$$

Example: $$\mathbb{H}^V(N) := \sum_{j=1}^{N} \left(\frac{-\Delta_{x_j}}{2} + V(x_j) \right) + \sum_{1\le k<\ell\le N} W(x_k - x_\ell)$$

2^{nd} quant.:

$$\mathbb{H}^V = \sum_{i,j} h_{ij}\, a^+(f_i)\, a(f_j) + \sum_{\substack{1\le k\le \ell \\ 1\le i\le j}} W_{ij,k\ell}\, a^+(f_i)\, a^+(f_j)\, a(f_\ell)\, a(f_k)$$

w/ $$h_{ij} := \int \left(\frac{\overline{\nabla f_i(x)} \cdot \nabla f_j(x)}{2} + V(x)\, \overline{f_i(x)}\, f_j(x) \right) dx .$$

Physicists' notation: $a^+(f) = \int_{\mathbb{R}^d} f(x)\, \varphi^+(x)\, dx \rightsquigarrow \varphi^+(x) = \sum_{i\ge 1} \overline{f_i(x)}\, a^+(f_i)$ creation op. at position $x \in \mathbb{R}^d$

w/ $\{f_i\}_i \subseteq \mathcal{H}$ an ONB.

$$\hookrightarrow \mathbb{H} = \int_{\mathbb{R}^d} \left(\frac{1}{2} \nabla\varphi^+(x) \cdot \nabla\varphi(x) + V(x)\, \varphi^+(x)\, \varphi(x) \right) dx + \frac{1}{2} \int_{\mathbb{R}^{2d}} W(x-y)\, \varphi^+(x)\, \varphi^+(y)\, \varphi(y)\, \varphi(x)\, d(x,y)$$

4) States, density matrices

$\mathcal{H}$ H-sp.

Def. (State). $\Gamma \in \mathfrak{S}_1(\mathcal{H})$ is a state iff $\Gamma \geq 0$ is s.a. and $\mathrm{Tr}(\Gamma) = 1$.

Denote the set of all states by $S(\mathcal{H})$.

Note that $S(\mathcal{H})$ is cvx and by the Spectral Thm, every state $\Gamma \in S(\mathcal{H})$ can be written as

$$\Gamma = \sum_{i \geq 1} u_i |\Psi_i\rangle\langle\Psi_i|, \quad \sum_{i \geq 1} u_i = 1.$$

$S(\mathcal{H})$ is not closed under weak-$*$ limit: $\Gamma_n \xrightarrow{*} \Gamma$, $(\Gamma_n)_n \in S(\mathcal{H})^{\mathbb{N}} \Rightarrow \Gamma = \Gamma^* \geq 0$ w/

$$\mathrm{Tr}_{\mathcal{H}}(\Gamma) \leq \liminf_{n \to \infty} \mathrm{Tr}_{\mathcal{H}}(\Gamma_n) = 1$$

There are examples where $<$ holds: Take $\Psi_n := \varphi_1 \wedge \varphi_n$ for ONS $\{\varphi_n\}$. $\Psi_n \rightharpoonup 0 \Rightarrow |\Psi_n\rangle\langle\Psi_n| \xrightarrow{*} 0$.

Def. (density matrix) Let $\Gamma \in S(\mathcal{F}_{a/s})$. Define $\Gamma^{(p,q)} : \mathfrak{H}^q_{a/s} \longrightarrow \mathfrak{H}^p_{a/s}$ by

$$\langle g_1 \{\substack{\wedge \\ \vee}\} \cdots \{\substack{\wedge \\ \vee}\} g_p, \Gamma^{(p,q)} f_1 \{\substack{\wedge \\ \vee}\} \cdots \{\substack{\wedge \\ \vee}\} f_q \rangle = \mathrm{Tr}_{\mathcal{F}_{a/s}}\left(\Gamma a^*(f_1) \ldots a^*(f_q) a(g_p) \ldots a(g_1)\right).$$

Write $\Gamma^{(p)} := \Gamma^{(p,p)}$. Have $\Gamma^{(0)} = \mathrm{Tr}(\Gamma) = 1$.

Properties. (i) $\forall N \in \mathbb{N}\ \exists C_N > 0\ \forall 0 \leq p, q \leq N\ \forall \Gamma \in S(\mathcal{F}_{a/s}^{\leq N})$: $\|\Gamma^{(p,q)}\|_{\underbrace{\mathfrak{S}_1(\mathfrak{H}^p_{a/s}, \mathfrak{H}^q_{a/s})}_{\text{Def. via singular values}}} \leq C_N$

(ii) $S(\mathcal{F}_{a/s}^{\leq N}) \ni \Gamma \longmapsto \Gamma^{(p,q)} \in \mathfrak{S}_1(\mathfrak{H}^q_{a/s}, \mathfrak{H}^p_{a/s})$ is continuous but not weakly-$*$ continuous.

(iii) States fully determined by density matrices: $\Gamma_1^{(p,q)} = \Gamma_2^{(p,q)}\ \forall p,q \Rightarrow \Gamma_1 = \Gamma_2$

$(\Gamma_1, \Gamma_2 \in S(\mathcal{F}_{a/s}^{\leq N}))$.

Rem.: States in this context generalize the concept of states in QM ($= L^2$-fcns w/ $\|\ \|_2 = 1$)

Had: $\mathcal{E}(\Psi) = \langle \Psi, H\Psi \rangle = \mathrm{Tr}(H|\Psi\rangle\langle\Psi|)$

$\leadsto$ Have $\mathcal{E}(\Gamma) = \mathrm{Tr}_{\mathcal{F}}(\underbrace{\mathbb{H}}_{2^{\text{nd}}\text{ quant. of } H}\Gamma)$ iff $\Gamma \in S(\mathcal{F}_{a/s}^{\text{fin}})$.

Geometric convergence

Def. (Geometric convergence)

Def. the *geometric topology* $\mathcal{T}_g$ on $S(\mathcal{F}^{\leq N})$ to be the initial topology for the maps

$$S(\mathcal{F}^{\leq N}) \ni \Gamma \longmapsto \langle \Phi, \Gamma^{(p,q)} \Psi \rangle_{\mathcal{H}^p}.$$

We say $(\Gamma_n)_n \in S(\mathcal{F}^{\leq N})^{\mathbb{N}}$ *converges geometrically* to $\Gamma \in \mathfrak{S}_1(\mathcal{F}^{\leq N})$ iff

$$\lim_{n\to\infty} \langle \Phi, \Gamma_n^{(p,q)} \Psi \rangle_{\mathcal{H}^p} = \langle \Phi, \Gamma^{(p,q)} \Psi \rangle \quad \forall \Phi, \Psi, p, q.$$

In this case, we write $\Gamma_n \xrightarrow[g]{} \Gamma$.

Properties: (i) $\mathcal{T}_g \overset{\text{coarser}}{\subseteq} \mathcal{T}_{\mathfrak{S}_1(\mathcal{F}^{\leq N})}$: $\Gamma_n \to \Gamma$ in $\mathfrak{S}_1(\mathcal{F}^{\leq N}) \Rightarrow \Gamma_n \xrightarrow[g]{} \Gamma$

(ii) $\Gamma_n \xrightarrow[g]{} \Gamma$ in $S(\mathcal{F}^{\leq N}) \Longleftrightarrow \Gamma_n^{(p,q)} \xrightarrow[*]{} \Gamma^{(p,q)}$ weakly-$*$ in $\mathfrak{S}_1(\mathcal{F}^{\leq N})$ $\forall 0 \leq p,q \leq N$.

(iii) Note that $S(\mathcal{F}^{\leq N}) \subseteq \mathfrak{S}_1(\mathcal{F}^{\leq N})$ is closed under $\mathcal{T}_g$ but not under the weak-$*$ topology.

Example: $\{\varphi_n\}_{n} \subseteq \mathcal{H}$ ONB. Def. $\Psi_n := \varphi_1 \wedge \varphi_n \rightsquigarrow \Gamma_n := 0 \oplus 0 \oplus |\Psi_n\rangle\langle\Psi_n| \in S(\mathcal{F}_a^{\leq 2})$.

Have $\Gamma_n \xrightarrow[*]{} 0$ weakly-$*$ but $\Gamma_n \xrightarrow[g]{} 0 \oplus |\varphi_1\rangle\langle\varphi_1| \oplus 0$.

(iv) Even more: $(S(\mathcal{F}^{\leq N}), \mathcal{T}_g)$ is compact, i.e., $(\Gamma_n)_n \in S(\mathcal{F}^{\leq N})^{\mathbb{N}}$ has a conv. subseq. $\Gamma_{n_k} \xrightarrow[g]{} \Gamma$

$\rightsquigarrow$ Follows from $\mathcal{T}_g$ being the restriction of a topology on the CAR (CCR) algebra generated by $\{a(f), a^*(g)\}$.

(v) $\left(\begin{array}{c}\text{Average particle number conservation} \\ + \\ \text{geometric convergence}\end{array}\right) \Longleftrightarrow$ strong convergence:

Geometric lsc: $\Gamma_n \xrightarrow[g]{} \Gamma \Rightarrow \mathrm{Tr}_{\mathcal{F}}(\mathcal{N}\Gamma) \leq \liminf_{n\to\infty} \mathrm{Tr}_{\mathcal{F}}(\mathcal{N}\Gamma_n)$

w/ "=" $\Leftrightarrow \Gamma_n \to \Gamma$

Application: HVZ Theorem

Recall. $$H^V(k) := \sum_{j=1}^{k}\left(-\frac{\Delta_{x_j}}{2}+V(x_j)\right)+\sum_{1\le k<\ell\le N} W(x_k-x_\ell)$$

and $$\mathbb{H}^V = 0 \oplus \bigoplus_{k\geq 1} H(k).$$

For a big class of V, W have
$$(1-\varepsilon)\left(\sum_{j=1}^{N}-\Delta_{x_j}\right)-C \le H^V(N) \le (1+\varepsilon)\left(\sum_{j=1}^{N}-\Delta_{x_j}\right)+C.$$

Assume this. If $\mathrm{Tr}\left((-\Delta)^{\frac{1}{2}}\Gamma^{(1)}(-\Delta)^{\frac{1}{2}}\right)<\infty$ define
$$\mathcal{E}^V(\Gamma) := \mathrm{Tr}_{\mathcal{F}_{a/s}}\left(\mathbb{H}^V\Gamma\right) = \mathrm{Tr}_{L^2}\left[\left(-\tfrac{1}{2}\Delta+V\right)\Gamma^{(1)}\right]+\mathrm{Tr}_{\mathfrak{H}^2_{a/s}}\left(W\Gamma^{(2)}\right).$$

Motivation: $\Psi \longmapsto \langle \Psi, H(N)\Psi\rangle$ is not weak lsc. Indeed, defining
$$E^V(N) := \inf \sigma(H^V(N)),\quad \Sigma^V(N) := \inf \sigma_{ess}(H^V(N))$$
typically have $\Sigma^V(N)<0$, i.e., $\exists$ singular Weyl sequence $\|\Psi_n\|=1,\ (H^V(N)-\Sigma^V(N))\Psi_n \xrightarrow[n\to\infty]{} 0$ $\Psi_n \rightharpoonup 0$ w/ $\langle \Psi_n, H^V(N)\Psi_n\rangle \to \Sigma^V(N)<0$.

Lemma: Assume $W\geq 0$, $(\Gamma_n)_n \in S(\mathcal{F}^{\le N})^{\mathbb{N}}$, $\Gamma_n \xrightarrow[g]{} \Gamma$. Then
$$\mathcal{E}^V(\Gamma) \le \liminf_{n\to\infty} \mathcal{E}^V(\Gamma_n).$$

Thm (**H**unziker **V**an Winter **Z**hislin in lsc case)

Let $W\geq 0$. Then $E^0(N)=0$ $\forall N\geq 0$ and $\Sigma^V(N)=E^V(N-1)$.

In particular, $E^V(N)$ is an isolated ev iff
$$E^V(N)<E^V(N-1) = \min\left\{E^V(N-k)+E^0(k),\ k\in\{1,\dots,N\}\right\}.$$

Remark: For $\Gamma \in S(\mathcal{F}^{\leq N})$ def. $G_{mn} := \Pi_m \Gamma \Pi_n : \mathcal{H}^n_{a/s} \to \mathcal{H}^m_{a/s}$.

If $[\Gamma, \mathcal{N}] = 0$, then

$$\Gamma = \begin{pmatrix} G_{11} & & 0 \\ & \ddots & \\ 0 & & G_{NN} \end{pmatrix}.$$

Now take $\Gamma_n \xrightarrow{*} \Gamma$ w/ $[\Gamma_n, \mathcal{N}] = 0$. Then $\Gamma_n^{(p,q)} = 0$ for $p \neq q$ and thus $\Gamma^{(p,q)} = 0$ for $p \neq q$.

Thus $[\Gamma, \mathcal{N}] = 0$.

Proof of Thm. $\Sigma^V(N) \leq E^V(N-1)$ is shown by a std argument constructing an appropriate singular Weyl sequence, using a Weyl sequence for $E^V(N-1)$.

Now notice $E^V(N) \underset{\substack{\uparrow \\ \text{inf over} \\ \text{bigger set}}}{\leq} \Sigma^V(N) \leq E^V(N-1) \Rightarrow N \mapsto E^V(N-1)$ non-increasing.

For $V=0$, $E^0(N) \geq 0$ $(W \geq 0)$, $E^0(1) = 0 \Rightarrow E^0(N) = 0$ $\forall N$.

Take a singular Weyl sequence $(\Psi_n)_n \in (\mathcal{H}^N)^{\mathbb{N}}$ for $\Sigma^V(N)$.

$$(H^V(N) - \Sigma^V(N))\Psi_n \xrightarrow[n\to\infty]{} 0, \quad \|\Psi_n\| = 1, \quad \Psi_n \to 0 \text{ weakly in } \mathcal{H}^N.$$

Def. $\Gamma_n = 0 \oplus \ldots \oplus 0 \oplus |\Psi_n\rangle\langle\Psi_n| \in S(\mathcal{F}^{\leq N})$.

Up to taking a subsequence, may assume $\Gamma_n \xrightarrow{*} \Gamma \in S(\mathcal{F}^{\leq N})$. By the above remark have

$$\Gamma = \begin{pmatrix} G_{00} & & 0 \\ & \ddots & \\ 0 & & G_{NN} \end{pmatrix}$$

Have $G_{NN} = G_{NN}^{(N)}$ is weak-$*$ limit of $|\Psi_n\rangle\langle\Psi_n|$, thus $G_{NN} = 0$ for $\Psi_n \to 0$.

Then

$$\Sigma^V(N) \overset{\text{Weyl seq}}{=} \lim_{n\to\infty} \mathcal{E}^V(\Gamma_n) \overset{\text{Lemma}}{\geq} \mathcal{E}^V(\Gamma) = \sum_{j=0}^{N-1} \mathrm{Tr}_{\mathcal{H}^j}(H^V(j) G_{jj}) \overset{\substack{\text{var. char.} \\ \text{of ev}}}{\geq} \sum_{j=0}^{N-1} \underbrace{E^V(j)}_{\geq E^V(N-1)} \mathrm{Tr}_{\mathcal{H}^j}(\underbrace{G_{jj}}_{\geq 0})$$

$$\underset{\substack{\uparrow \\ \sum_{j=0}^{N-1} \mathrm{Tr}_{\mathcal{H}^j}(G_{jj}) = \mathrm{Tr}\,\Gamma = 1}}{\geq} E^V(N-1).$$

$\boxtimes$