Research statement - Michael Hott

My research field is the mathematical analysis of problems arising in physics, especially Statistical Physics, Quantum Field Theory, and Quantum Mechanics. My work merges methods of Renormalization, Functional, Geometric, and Harmonic Analysis, PDEs, and Spectral Theory.

My research focuses on the dynamics of large, complex systems of interacting quantum mechanical particles. The mathematically rigorous analysis of properties of gases of bosons undergoing Bose-Einstein condensation is an extremely active and important area in mathematical physics, especially following the 2001 Nobel Prize to Ketterle and Cornell-Wiemann for the first experimental verification. Fundamental results in the mathematical literature were obtained by Bogoliubov, and more recently by Lieb, Seiringer, Yngvason and their collaborators in the time-independent case, see [50, 51, 52, 53], and by Erdös, Schlein, Yau in the dynamical case, see also [30, 26, 27, 28, 29, 66]. These works provide a rigorous derivation of the nonlinear Hartree and nonlinear Schrödinger equation

$$i\partial_t\phi \;=\; -\Delta\phi + (v * |\phi|^2)\phi\,, \tag{1}$$

where ϕ describes the Bose-Einstein condensate (BEC), and v is the pair interaction potential. Fluctuations around the BEC obey the Hartree-Fock-Bogoliubov (HFB) dynamics, see [6, 7, 37, 38, 39, 40].

Another extremely active research area intersecting Analysis and Physics is the derivation of nonlinear Boltzmann equations in classical collisonal systems, for instance of gases of hard spheres. Recent years have witnessed various breakthrough results, due to Bodineau, Gallagher, Saint-Raymond and many others, see [2, 3, 11, 15, 22, 33, 61, 63, 62] extending classical fundamental works of Cercignani, Lanford, DiPerna-Lions, and others, see [14, 16, 23, 46, 48, 68, 67, 73].

Derivation of Quantum Boltzmann equations. One question that has remained open for decades in the community has been whether nonlinear Boltzmann equations do emerge from complex interacting quantum systems. Preliminary works both in the physics literature [1, 41, 44, 72, 74], and in the mathematical literature [8, 9, 10, 18, 25] suggest that this is conjecturally true.

In my work [17] with Thomas Chen, we prove rigorously that, near a BEC, the quantum fluctuations subleading to the HFB dynamics is of quantum Boltzmann type. More precisely, we study the fluctuation dynamics of a Bose gas in the presence of a BEC with particle density N, trapped in a 3-torus $\Lambda = [-L/2, L/2]^3/\sim$ of linear size $L \geqslant 1$, with Hamiltonian

$$\int_\Lambda dx\, a_x^\dagger\big(-\frac{1}{2}\Delta_x \,+\, w(x)\big)a_x + \frac{\lambda}{2N}\int_{\Lambda^2} dx\, dy\, v(x-y)\, a_x^\dagger a_y^\dagger a_y a_x\,, \tag{2}$$

acting on the bosonic Fock space $\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n\geqslant 1}(L^2(\Lambda))^{\otimes_s n}$. Here w is a sufficiently regular external trapping potential, $\lambda > 0$, and $\hat{v} \geqslant 0$ is radial and sufficiently regular. a_x, $a_y^\dagger$ are bosonic annihilation and creation operators on Λ satisfying canonical commutation relations (CCR). The leading order dynamics, describing the BEC with L^2-density N, is given by a Hartree equation, as we noted above. The leading order fluctuation dynamics is described by the Hartree-Fock-Bogoliubov (HFB) equations. They describe the dynamics of fluctuation particles absorbed or emitted by the BEC $\sqrt{N|\Lambda|}\phi_t$.

It has been a longstanding open problem to show that fluctuations around the BEC exhibit a Boltzmann type behavior, of the form

$$\partial_T F_T \;=\; Q(F_T)\,, \tag{3}$$

where $F_T(p) = \frac{1}{|\Lambda|}\langle a_p^\dagger a_p\rangle_T$ for a suitable class of time-dependent states $\langle\cdot\rangle_T$ and Q is a Boltzmann-type collision term. Here, $T = t\lambda^2$ denotes macroscopic time, and t denotes microscopic time. In our work [17], we prove a result of this type; notably our analysis shows that there are various subtleties in this problem going beyond the standard formulations of this conjecture.

We have that the density of HFB fluctuation particles is of order $O(1)$ and it evolves at frequency $O(1)$ as $N \gg 1$ and $\lambda \ll 1$. In order to analyze lower order fluctuations, we subtract both the BEC dynamics and the HFB dynamics from the full dynamics defined by the Hamiltonian (2), and we denote the resulting time-evolved state by $\langle\cdot\rangle_t$. We choose a static translation invariant solution $\phi_0 = \phi_t = 1/\sqrt{|\Lambda|}$ of the Hartree equation. As an initial state $\langle\cdot\rangle_0$, we consider a uniformly distributed BEC of density N with excited particles described by a regularized quasi-free Gibbs state that commutes with the number operator. We show that there exists $\lambda = \lambda(N)$ such that the following holds for large N. Let $\Psi_T := \langle a_0\rangle_{T\lambda^{-2}}/|\Lambda|$, $F_T(p) := (\langle a_p^\dagger a_p\rangle_{T\lambda^{-2}} - |\Psi_T\delta(p)|^2)/|\Lambda|$ and $G_T := (\langle a_p a_{-p}\rangle_{T\lambda^{-2}} - (\Psi_T\delta(p))^2)/|\Lambda|$ be the centered 1-point and 2-point mescoscopic correlation functions, where $\delta(p) := |\Lambda|\delta_{p,0}$ on the lattice $\Lambda^* = (\frac{2\pi}{L}\mathbb{Z})^3$. In the case $|\Lambda| \sim \lambda^{-7}$, we show that the correlations at kinetic time $t = T/\lambda^2$ with $\lambda = \lambda(N)$ satisfy the dynamics

$$\Psi_T = \frac{i}{\lambda\sqrt{N}} \int_0^T dS \int_{\mathbb{R}^3} \hat{v} F_S + l.o.t., \tag{4}$$

$$F_T - F_0 = \frac{1}{N} \int_0^T dS\, Q^{(mol)}(F_S, F_S, F_S) + l.o.t., \tag{5}$$

$$G_T = \frac{1}{N} \mathcal{A}(F, F)(T) + \frac{1}{N\lambda^2} \mathcal{Q}(F, F, F)(T) + l.o.t., \tag{6}$$

where the Boltzmann collision term $Q^{(mol)}$ is a mollification of the expression predicted in the literature [64, 74]. Denoting $\widetilde{F_S} := 1 + F_S$ and the Bogoliubov dispersion $\Omega := \sqrt{E(E + 2\lambda\hat{v})}$, $E = |p|^2/2$, it is of the form

$$\begin{aligned} Q^{(mol)}(F_S, F_S, F_S)(p) := \quad & \int_0^T dS \int_{(\mathbb{R}^3)^2} dp_1\, dp_2\, \frac{\sin\left(\frac{T-S}{\lambda^2}(\Omega(p_1) + \Omega(p_2) - \Omega(p_1 + p_2))\right)}{\Omega(p_1) + \Omega(p_2) - \Omega(p_1 + p_2)} \\ & (\hat{v}(p_1) + \hat{v}(p_2))^2 \big(\delta(p - p_1) + \delta(p - p_2) - \delta(p - (p_1 + p_2))\big) \\ & \big(\widetilde{F_S}(p_1)\widetilde{F_S}(p_2)F_S(p_1 + p_2) - F_S(p_1)F_S(p_2)\widetilde{F_S}(p_1 + p_2)\big). \end{aligned} \tag{7}$$

In the case $|\Lambda| \sim 1$, integrals over $\mathbb{R}^3$ are not well approximated by sums over Λ^*, and we show that the Boltzmann collision operator contains subleading terms that can become dominant, depending on time-dependent coefficients assuming particular values in $\mathbb{Q}$; this phenomenon is reminiscent of the Talbot effect [24, 45]. This phenomenon also occurs for the dynamics of G.

Our work introduces the concept of approximate restricted quasi-freeness (ARQF), which controls an approximate Wick Theorem for polynomials in $a^\#$ up to a given order. Employing a propagation of moments result in [12, 65], we show that ARQF is propagated, providing quantitative bounds, which resembles the notion of propagation of chaos in kinetic theory.

Derivation of Hartree equation In [4, 5, 42], we studied the derivation of Hartree equations for non-relativistic, relativistic and fractional dynamical bosons with singular interactions

$$i\partial_t\phi_t = (-\Delta)^\sigma \phi_t + \frac{\lambda}{|\cdot|^\gamma} * |\phi_t|^2\phi_t. \tag{8}$$

We simplified previous arguments, extended the parameter range, improved convergence rates, and showed convergence in a coarser topology, allowing to consider unbounded observables. We also extended the analysis to mixtures of BECs.

Outlook The derivation of the quantum Boltzmann equation from an interacting Bose gas has been an open problem for decades that was at the center of interest in this area of research. Our result is a first step towards a more thorough understanding of this problem, and provides many directions of refinements and extensions.

Partially joint with my collaborator Thomas Chen and partially in solo projects, I plan to use Renormalization ideas from Quantum Field Theory (QFT) to increase the time of validity of the Boltzmann equation. This will likely add quantum corrections to the quantum Boltzmann dynamics. Our current work addresses a specific setup when, in a Boson gas at low temperatures, a large BEC has already formed and derives the effective dynamics for a fixed short period of time. I intend to investigate processes at lower and higher condensate density, in order to understand the long-time behavior of Bose gases at or below condensation temperature. It is also interesting to study whether Boltzmann collisions can be observed above the condensation temperature and for how long. Another anticipated direction is to find rigorous justifications of other models studied in applied mathematics and PDEs describing quantum Boltzmann processes, see, e.g., [60, 64, 70, 74]. Crucially for this work, I intend to deepen my expertise developing tools merging techniques from QFT, Harmonic Analysis, Geometric Measure Theory, and nonlinear PDE for a deeper understanding of the dynamics of large complex quantum systems.

I would like to extend my research portfolio with deeper results, and make original and impactful contributions in the vibrant intersecting area of Mathematical Physics, Kinetic Theory, Analysis and Applied Mathematics. Ultimately, I aim to pursue an academic career as a tenure stream faculty member at a highly research active institution, continuing to build a network of strong research relationships, and help students and young researchers on their careers.

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