SAMPLE SOLUTIONS AND GRADES M 362K, Fall 03

Here are two problems and some sample solutions, with the grades (out of a maximum possible 10) that they would receive.

Problem 1: Find $\int_{-\infty}^{\infty} xe^{-x^2}dx$.

Solution I: *(This solution would receive 10 points.)*

First find the indefinite integral $\int xe^{-x^2}dx$ by substitution. Let u = $-x^2$. Then du = - 2xdx, so xdx = -du/2. Thus

$$\int xe^{-x^2}dx = \int \frac{-e^u}{2}du = \frac{-e^u}{2} + C = \frac{-e^{-x^2}}{2} + C.$$

Now

$$\begin{aligned}
\int_{-\infty}^{\infty} xe^{-x^2}dx &= \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx \\
&= \lim_{a\to-\infty}\int_{a}^{0} xe^{-x^2}dx + \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2}dx \\
&= \lim_{a\to-\infty}\left[\frac{-e^{-x^2}}{2}\right]_a^0 + \lim_{b\to\infty}\left[\frac{-e^{-x^2}}{2}\right]_0^b \\
&= \lim_{a\to-\infty}\left[-\frac{1}{2} - \frac{-e^{-a^2}}{2}\right] + \lim_{b\to\infty}\left[\frac{-e^{-b^2}}{2} - \frac{-1}{2}\right] \\
&= -1/2 + 0 + 0 + 1/2 = 0
\end{aligned}$$

Solution II: *(This solution is like Solution I, except the student has not done the substitution explicitly. Since the substitution is simple enough that some students could do it in their heads, this solution would also receive 10 points. <u>However, for a more complicated substitution, you will not receive full credit if you do not show the substitution clearly</u>.)*

$$\begin{aligned}
\int_{-\infty}^{\infty} xe^{-x^2}dx &= \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx \\
&= \lim_{a\to-\infty}\int_{a}^{0} xe^{-x^2}dx + \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2}dx \\
&= \lim_{a\to-\infty}\left[\frac{-e^{-x^2}}{2}\right]_a^0 + \lim_{b\to\infty}\left[\frac{-e^{-x^2}}{2}\right]_0^b \\
&= \lim_{a\to-\infty}\left[-\frac{1}{2} - \frac{-e^{-a^2}}{2}\right] + \lim_{b\to\infty}\left[\frac{-e^{-b^2}}{2} - \frac{-1}{2}\right] \\
&= -1/2 + 0 + 0 + 1/2 = 0
\end{aligned}$$

Solution III: *(This solution would receive 10 points.)*

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx$$
$$= \lim_{a\to-\infty}\int_{a}^{0} xe^{-x^2}dx + \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2}dx$$

Let u = $-x^2$. Then du = - 2xdx, so xdx = -du/2. When x = a, u = $-a^2$. When x = 0, u = 0. When x = b, u = $-b^2$. So

$$\lim_{a\to-\infty}\int_{a}^{0} xe^{-x^2}dx + \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2}dx$$
$$= \lim_{a\to-\infty}\int_{-a^2}^{0}\left(-\frac{1}{2}e^u du\right) + \lim_{b\to\infty}\int_{0}^{-b^2}\left(-\frac{1}{2}e^u du\right)$$

$$= \lim_{a\to-\infty}\left[\frac{-e^u}{2}\right]_{-a^2}^{0} + \lim_{b\to\infty}\left[\frac{-e^u}{2}\right]_{0}^{-b^2}$$
$$= \lim_{a\to-\infty}\left[-\frac{1}{2}-\frac{-e^{-a^2}}{2}\right] + \lim_{b\to\infty}\left[\frac{-e^{-b^2}}{2}-\frac{-1}{2}\right]$$

= -1/2 + 0 + 0 + 1/2 = 0

Solution IV: *(This solution would receive 6 points. The student has forgotten to change the limits of integration in making the change of variables, and consequently gets the wrong answer.)*

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx$$
$$= \lim_{a\to-\infty}\int_{a}^{0} xe^{-x^2}dx + \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2}dx$$

Let u = $-x^2$. Then du = - 2xdx, so xdx = -du/2, so

$$\lim_{a\to-\infty}\int_{a}^{0} xe^{-x^2}dx + \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2}dx$$
$$= \lim_{a\to-\infty}\int_{a}^{0}\left(-\frac{1}{2}e^u du\right) + \lim_{b\to\infty}\int_{0}^{b}\left(-\frac{1}{2}e^u du\right)$$

$$= \lim_{a\to-\infty}\left[\frac{-e^u}{2}\right]_{a}^{0} + \lim_{b\to\infty}\left[\frac{-e^u}{2}\right]_{0}^{b}$$
$$= \lim_{a\to-\infty}\left[-\frac{1}{2}-\frac{-e^{a}}{2}\right] + \lim_{b\to\infty}\left[\frac{-e^{b}}{2}-\frac{-1}{2}\right]$$
$$= -1/2 + 0 + \lim_{b\to\infty}\left[\frac{-e^{b}}{2}\right] + 1/2\ .$$

Since $\lim_{b\to\infty}\left[\frac{-e^{b}}{2}\right] = \infty$, the entire integral diverges.

Solution V: *(This solution would receive 6 points. The final answer is correct, but the method has a serious error. I hope you can spot it!)*

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \lim_{a\to\infty}\int_{-a}^{a} xe^{-x^2}\,dx$$

Let u = $-x^2$. Then du = - 2xdx, so xdx = -du/2. When x = - a, u = $-a^2$. When x = a, u = $-a^2$. So

$$\lim_{a\to\infty}\int_{-a}^{a} xe^{-x^2}\,dx = \lim_{a\to\infty}\int_{-a^2}^{a^2}\left(-\frac{1}{2}e^u du\right)$$

$$= \lim_{a\to\infty}\left[\frac{-e^u}{2}\right]_{-a^2}^{a^2}$$

$$= \lim_{a\to\infty}\left[\frac{-e^{a^2}}{2} - \frac{-e^{-a^2}}{2}\right]$$

= 0 + 0 + = 0

Solution VI: *(This solution would receive 10 points. Note that the last part of this solution is not applicable to all improper integrals; it uses special properties of this particular problem.)*

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx$$

Let u = $-x^2$. Then du = - 2xdx, so xdx = -du/2. As x $\to$ -∞, u $\to$ - ∞. When x = 0, u = 0. As x $\to$ ∞, u $\to$ - ∞. So

$$\int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx$$

$$= \int_{-\infty}^{0}\left(-\frac{1}{2}e^u du\right) + \int_{0}^{-\infty}\left(-\frac{1}{2}e^u du\right)$$

$$= \int_{-\infty}^{0}\left(-\frac{1}{2}e^u du\right) - \int_{-\infty}^{0}\left(-\frac{1}{2}e^u du\right)$$

= 0

Solution VII: *(This solution would receive 10 points. Of course, this method is not applicable to all improper integrals; it depends on the special properties of this particular problem.)*

The function xe^{-x^2} is an odd function, so $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$ = 0.

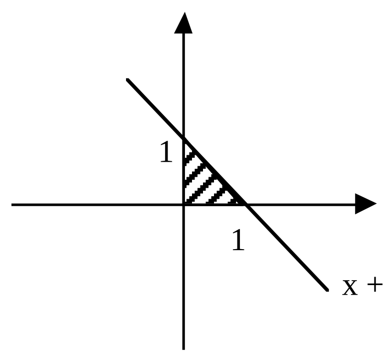
Solution VII: (*This solution is very much like some computations in the textbook. However, just as you are expected to fill in details missing in textbook computations, <u>you are expected to provide more detail in your solutions than the textbook does</u>, so this solution would receive only 6 points.)*

$$\int_{-\infty}^{\infty} xe^{-x^2} dx = \left.\frac{-e^{-x^2}}{2}\right|_{-\infty}^{\infty} = 0 - 0 = 0$$

Problem 2**:** Integrate the function f(x,y) = $\begin{cases} e^{-x-2y} & if \;\; x > 0 \; and \; y > 0 \\ 0 & otherwise \end{cases}$

over the region R described by x + y ≤ 1.

Solution I: (*This solution would receive 10 points.)* Since f(x, y) = 0 except when x > 0 and y > 0, $\iint_R f(x,y)dxdy = \iint_D f(x,y)dxdy$, where D is the region described by: x + y ≤ 1, x > 0, and y > 0. The region D is shaded in the following diagram:



From the diagram, we can see that D can be described as: x ≤ 1 - y, for 0 ≤ y ≤ 1. This means we can set up the integral, integrating with respect to x first, as

$$\iint_D f(x,y)dxdy = \int_0^1 \int_0^{1-y} e^{-x-2y} dxdy$$

$$= \int_0^1 e^{-2y} \left(\int_0^{1-y} e^{-x} dx \right) dy$$

$$= \int_0^1 e^{-2y} \left[e^{-x} \right]_0^{1-y} dy$$

$$= \int_0^1 e^{-2y} \left[e^{-(1-y)} - 1 \right] dy$$

$$= \int_0^1 \left(e^{-y-1} - e^{-2y} \right) dy$$

$$= \left[-e^{-y-1} + \frac{e^{-2y}}{2} \right]_0^1$$

$$= \left(-e^{-2} + \frac{e^{-2}}{2} \right) - \left(-e^{-1} + \frac{1}{2} \right)$$

$$= -\frac{e^{-2}}{2} + e^{-1} - \frac{1}{2}$$

Note: This problem could also be set up integrating with respect to y first.

Solution II: (*This solution would receive 8 points. It uses the same method as Solution I, but the explanation of how the integral is set up is missing.)*

$$\iint_R f(x,y)dxdy = \int_0^1 \int_0^{1-y} e^{-x-2y} dxdy$$

$$= \int_0^1 e^{-2y} \left(\int_0^{1-y} e^{-x} dx \right) dy$$

$$= \int_0^1 e^{-2y} \left[e^{-x} \right]_0^{1-y} dy$$

$$= \int_0^1 e^{-2y} \left[e^{-(1-y)} - 1 \right] dy$$

$$= \int_0^1 \left(e^{-y-1} - e^{-2y} \right) dy$$

$$= \left[-e^{-y-1} + \frac{e^{-2y}}{2} \right]_0^1$$

$$= \left(-e^{-2} + \frac{e^{-2}}{2} \right) - \left(-e^{-1} + \frac{1}{2} \right)$$

$$= -\frac{e^{-2}}{2} + e^{-1} - \frac{1}{2}$$