ANALYSIS OF BALANCED FACTORIAL DESIGNS

Estimates of model parameters and contrasts can be obtained by the method of Least Squares. Additional constraints must be added to estimate non-estimable parameters.

Example: The cell means are estimable with Least Squares estimates $\bar{y}_{ijk\bullet}$.

From the cell-means model and one-way analysis of variance, we have the following:

- The *error sum of squares* ssE is the sum of squared deviations from the fits $\bar{y}_{ijk\bullet}$.
  Example: For 3 factors, ssE = $\sum_i \sum_j \sum_k \sum_t (y_{ijkt} - \bar{y}_{ijk\bullet})^2$
- ssE has associated degrees of freedom n – v
  Example: For 3 factors, ssE has abcr – abc = abc(r-1) degrees of freedom.
- The *mean square error* msE is ssE divided by its degrees of freedom.
  Example: For 3 factors, msE = ssE/ abc(r-1) degrees of freedom.
- msE is an estimate for the variance σ^2.
- The standard error of the residuals $y_{ijkt} - \bar{y}_{ijk\bullet}$ is ssE/(n-1)

Before doing inference, we should use the fits and residuals to check the model assumptions of independent, normal errors with constant variance.

Example: Evaporation experiment
The experimenters wanted to study the evaporation rate of water under different conditions. They varied three factors:

Factor	*Levels*
A: Container	1: cup 2: plate
B: Salt concentration	1: 1 tsp salt per 100 cc water 2: 3 tsp salt per 100 cc water
C: Location	1: windowsill 2: floor in closet

After 48 hours in the location, the amount of water remaining in the container was measured. The response was amount evaporated (= 100 cc – amount remaining).
Two observations were taken for each treatment combination.

Look at the data thinking about estimating standard deviations of errors for treatment combinations. What do you notice that needs thought?

Use the cell-means model to obtain residuals and test for equal variance. What requires thought?

Compare standard deviations of residuals between levels of each factor:

Factor	Level 1	Level 2	Estimated variance ratio
A	0.518	0.327	2.5
B	0.2673	0.551	4.25
C	0.423	0.443	1.1

Should we proceed to further model checks or not?

Simulation: Ten samples of eight each from a normal distribution with mean 0, standard deviation 0.4.

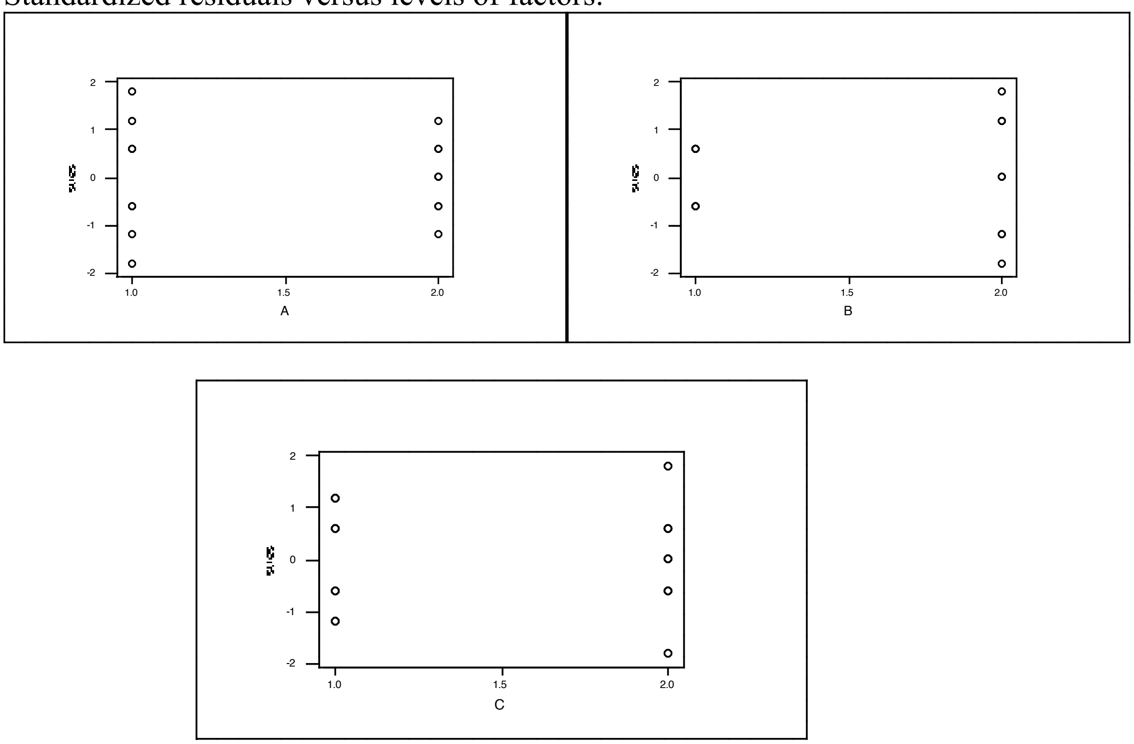
Standard deviations of the samples:

0.458 0.303 0.386 0.240 0.306 0.386 0.422 0.311 0.319 0.594

These would give an "estimated ratio of max to min variance" of 6.13.

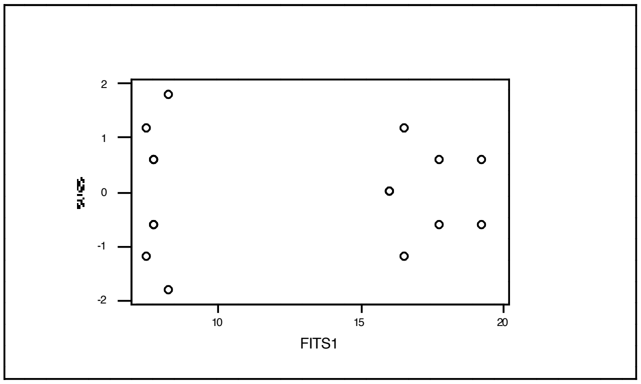
So we will proceed, but with some caution. First, model checking plots:

Standardized residuals versus levels of factors:



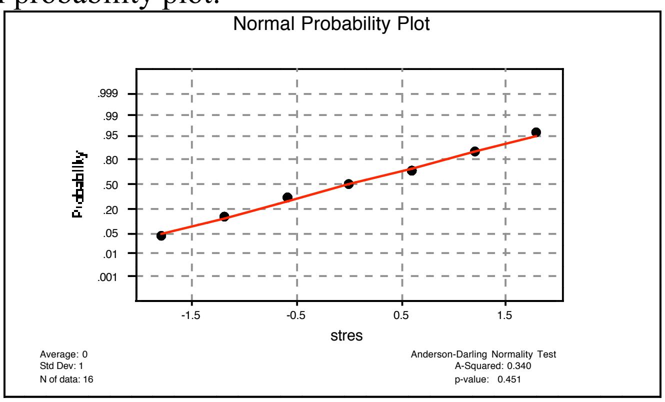
Does anything look problematical? If so, discuss.

Versus fits:



Any problems?

Normal probability plot:



Any problems?

Hypothesis Tests and Analysis of Variance Table:

Submodels can be tested against larger models by F-tests, with F-statistic obtained as a ratio of mean squares.

The *error sum of squares* of a model is the sum of the squared deviations from the fits from that model.

Example: For three factors, the error sum of squares for a model is $\sum_i \sum_j \sum_k \sum_t (y_{ijkt} - \hat{y})^2$, where $\hat{y}$ is the fit for the model.

Special case I: Since the fit for the complete model is $\bar{y}_{ijk\bullet}$, the error sum of squares for the complete model is ssE mentioned above.

Special case II: The *total sum of squares* is the error sum of squares for the model $y_{ijkt} = \mu + \varepsilon_{ijkt}$

Since the fit for this model is the overall mean, the total sum of squares can be described as the sum of the squared deviations from the overall mean.

Example: For 3 factors, sstot = $\sum_i \sum_j \sum_k \sum_t (y_{ijkt} - \bar{y}_{....})^2$

The total sum of squares has n-1 associated degrees of freedom, where n = total number of observations (e.g., abcr for 3 factors).

The sum of squares for a main effect or interaction is the difference between the error sum of squares for the model excluding terms involving this effect or interaction from the full model and the full model.

Example: For 3 factors, ssBC = ssE_0-ssE_a , where ssE_0 is the error sum of squares for the model excluding terms for BC and ABC interactions and ssE_a = ssE is the error sum of squares for the complete 3-way model.

The degrees of freedom for a sum of squares for a main effect or interaction is the product of one less than the number of levels for each factor involved in the effect or interaction.

Examples:

Sum of squares for	Degrees of freedom
A	a-1
AB	(a-1)(b-1)
BC	(b-1)(c-1)
ABC	(a-1)(b-1)(c-1)

The total sum of squares partitions into (is the sum of) the error sum of squares plus the sums of squares for all main effects and interactions in the model.

The degrees of freedom add correspondingly. For example, for the complete 3-way model, this is just the algebraic identity

$$(a-1)(b-1)(c-1) + (a-1)(b-1) + (a-1)(c-1) + (b-1)(c-1) + (a-1) + (b-1) + (c-1) + abcr - abc = abcr - 1$$

Note: If we are dealing with another model, the error sum of squares for that model will have a different number of degrees of freedom. The degrees of freedom for error are usually most easily found by subtracting the other degrees of freedom from the total degrees of freedom.

The *mean square* for a main effect or interaction is the sum of squares for that effect/interaction divided by its degrees of freedom.

Example: msBC = ssBC/(b-1)(c-1)

The F-statistic for a main effect or interaction is the mean square for the effect/interaction divided by the mean square error, with corresponding degrees of freedom in the numerator in denominator.

Example: In the 3-way complete model, the null hypothesis for BC interaction is

H_0: $[(\beta\gamma)_{ij} - (\beta\gamma)_{iq}] - [(\beta\gamma)_{sj} - (\beta\gamma)_{sq}] = 0$ for all $i \neq s$, $j \neq q$
and an analogous condition on the $\alpha\beta\gamma$ terms

The F-statistic for this test is F = msBC/msE, which has an F((b-1)(c-1), abc(r-1)) distribution if H_0 is true.

Example: Evaporation data.
Check out degrees of freedom, etc.
Conclusions? (Take into account overall significance level.)

Contrasts
Contrasts are defined and estimated as in the cell means model. There are various labels that are given to certain types of contrasts; see p. 199 for details. Contrasts can also be defined relative to submodels, in which case parameters not present in the submodel are omitted from the contrast. The various methods for simultaneous confidence intervals still apply. There is a modification that can be used for finding simultaneous confidence intervals for contrasts in the levels of a single factor: Replace v by the number of levels of the factor in question, and replace r by the number of observations on each level of the factor of interest. (See p. 205 for an example.)

Example: Suppose that in the evaporation experiment, the researchers were interested in comparing the levels of salt for all combinations of levels of the other two factors. Thus they were considering the 4 contrasts $\tau_{i1k} - \tau_{i2k}$. (What does a positive contrast mean in terms of the context of the experiment?)If they wanted simultaneous 98% confidence intervals, they might consider the following methods:
1. The Tukey method for all contrasts, giving msd 2.69 (from Minitab output)
2. The Bonferroni method, which would give individual 99.5% confidence intervals. The msd for this method is t(16 – 8, .0025) $\sqrt{msE\left(\frac{1}{2}+\frac{1}{2}\right)}$ = 3.8325 $\sqrt{0.328}$ = 1.904.
3. The Scheffe method, giving msd $\sqrt{(8-1)F_{8-1,16-8,.02}}\sqrt{msE\left(\frac{1}{2}+\frac{1}{2}\right)} =$

$[7(4.8972)]^{1/2}\sqrt{0.328} = 3.35$.

Bonferroni is clearly the best in this situation. The estimates and resulting simultaneous 98% confidence intervals are:

i,k	**Estimate for $\tau_{i1k} - \tau_{i2k}$**	**CI**
11	7.750 -7.500 =0.250	(-1.654, 2.154)
12	7.750 - 8.250 = - 0.50	(-2.404, 1.404)
21	17.750 - 16.500 = 1.25	(-0.625, 3.154)
22	19.250 – 16.000 = 3.250	(1.346, 5.154)

Thus, at an overall 98% confidence level, if the model assumptions indeed fit, we see that the higher concentration solution evaporates at a rate that was statistically significantly lower only when the container was the saucer (high surface area) and the location was the closet (low air movement).