TRANSFORMATIONS TO OBTAIN EQUAL VARIANCE

General method for finding variance-stabilizing transformations: If Y has mean μ and variance σ^2, and if U = f(Y), then by the first order Taylor approximation,

$$U \approx f(\mu) + (Y - \mu) f'(\mu),$$

so

$$\begin{aligned} Var(U) &\approx Var[f(\mu) + (Y - \mu) f'(\mu)] \\ &= [f'(\mu)]^2 Var(Y - \mu) \\ &= [f'(\mu)]^2 \sigma^2. \end{aligned}$$

If we have an ANOVA situation in which the group variances σ_i^2 are a function of the group means μ_i, say

$$\sigma_i^2 = g(\mu_i),$$

then if we choose the function f so that

$$f'(y) = [g(y)]^{-1/2},$$

and take $U_i = f(Y_i)$ (where Y_i denotes the response variable), we will have

$$\begin{aligned} Var(U_i) &\approx [f'(\mu_i)]^2 \sigma_i^2. \\ &= [g(\mu)]^{-1} g(\mu)] = 1. \end{aligned}$$

Thus such a transformation (or any scalar multiple of it) should give approximately equal variance.

Example: If $\sigma_i^2 \approx k(\mu_i)^q$, then $g(y) = ky^q$, so we want $f'(y) \propto y^{-\frac{q}{2}}$, giving

$$f(y) \propto \begin{cases} y^{1-\frac{q}{2}}, if \ q \neq 2 \\ \ln(y), if\, q = 2 \end{cases}.$$

(If some of the y's are zero or negative, then we will need to add a suitable constant to y before taking a negative power or log.)

Often a suitable value of q can be determined empirically, using the following idea:
If $\sigma_i^2 \approx k(\mu_i)^q$, then $\ln(\sigma_i^2) \approx \ln(k) + q\ln(\mu_i)$, so

- If a plot of $\ln(\sigma_i^2)$ vs $\ln(\mu_i)$ is close to a straight line, then a power transformation is a suitable choice.
- In this event, q can be estimated as the slope of a line approximately fitting this plot.

Cautions:

- Other model assumptions (especially normality) need to be checked before running the analysis, since the transformation might mess up other assumptions.
- Significance levels and confidence levels using transformed data will only be approximate, since the model has been changed *based on the data*.
- Interpretations need to be made in terms of the transformed units, or transformed back to the original units with care not to misinterpret.

Example: Battery data, with response "battery life" (rather than life per dollar).

Transformations based on theoretical considerations: Sometimes theoretical considerations point to a particular relationship between mean and variance, suggesting a particular transformation. Examples:

Type of Distribution	**Mean/Variance relationship**	**Type of Transformation**	**Comments**
Poisson	Variance = mean (so q = 1)	Square root (1-q/2 = 1/2)	1. Likely to occur with count data for rare events -- e.g., counts of accidents, flaws, or contaminating particles. 2. Simulations suggest that for sample size 15, the transformation does not substantially alter the probability of false rejection.
Binomial	Mean = mp, variance = mp(1-p)	$\arcsin\left(\sqrt{\frac{y}{m}}\right)$	1. Likely to occur with count data such as number of seeds in a fixed number that germinate, number of culture plates that grow visible bacteria colonies. 2. Simulations suggest that for m = 10, transformation does not change probability of false rejection.
Exponential	Variance = mean^2 (q = 2)	Log(y) (1 - q/2 = 0)	1. Likely to occur with certain kinds of reaction times, waiting times, and financial data. 2. Simulations suggest that with small sample sizes and differences in group means is large, transformation increase power, but in other cases can decrease power.