INFERENCE FOR CONTRASTS (Chapter 4)

Recall: A *contrast* is a linear combination of effects with coefficients summing to zero:

$$\sum_{i=1}^{v} c_i \tau_i \text{ where } \sum_{i=1}^{v} c_i = 0.$$

Specific types of contrasts of interest include:

- Differences in effects
- Differences in means

A special type of difference in means is often of interest in an experiment with a control group: The difference between the control group effect and the mean of the other treatment effects.

Recall that the least squares estimator of the contrast $\sum_{i=1}^{v} c_i \tau_i$ is $\sum_{i=1}^{v} c_i \hat{\tau}_i = \sum_{i=1}^{v} c_i \overline{Y}_{i\cdot}$

This is an unbiased estimator of the contrast:

$$\mathrm{E}(\sum_{i=1}^{v} c_i \overline{Y}_{i\cdot}) = \sum_{i=1}^{v} E(c_i \overline{Y}_{i\cdot}) = \sum_{i=1}^{v} c_i E(\overline{Y}_{i\cdot}) = \sum_{i=1}^{v} c_i(\mu + \tau_i) = \sum_{i=1}^{v} c_i \mu + \sum_{i=1}^{v} c_i \tau_i = \sum_{i=1}^{v} c_i \tau_i,$$

since $\sum_{i=1}^{v} c_i \mu = \left(\sum_{i=1}^{v} c_i\right)\mu = 0.$

Recall two of our model assumptions:

- $Y_{it} = \mu + \tau_i + \varepsilon_{it}$.
- The ε_{it} are independent random variables.

It follows that the Y_{it}'s are independent. Since each $\overline{Y}_{i\cdot}$ is a linear combination of the Y_{it}'s for the i[th] treatment group only, it follows that the $\overline{Y}_{i\cdot}$'s are independent. Thus

$$\mathrm{Var}(\sum_{i=1}^{v} c_i \overline{Y}_{i\cdot}) = \sum_{i=1}^{v} c_i^2 Var(\overline{Y}_{i\cdot}) = \sum_{i=1}^{v} c_i^2 \frac{\sigma^2}{r_i} = \sigma^2 \sum_{i=1}^{v} \frac{c_i^2}{r_i}.$$

Recall two of our model assumptions:

- $Y_{it} = \mu + \tau_i + \varepsilon_{it}$.
- For each i and t, $\varepsilon_{it} \sim N(0, \sigma^2)$

It follows from these that

$$Y_{it} \sim N(\mu + \tau_i, \sigma^2)$$

Since the Y_{it}'s are independent, each $\overline{Y}_{i\cdot}$, as a linear combination of independent normal random variables, is also normal. Since the contrast estimator $\sum_{i=1}^{v} c_i \overline{Y}_{i\cdot}$ is a linear combination of the independent normal random variables $\overline{Y}_{i\cdot}$, it too must be normal.

Summarizing:

$$\sum_{i=1}^{v} c_i \overline{Y}_{i\cdot} \sim \mathrm{N}\left(\sum_{i=1}^{v} c_i \tau_i, \sigma^2 \sum_{i=1}^{v} \frac{c_i^2}{r_i}\right).$$

Standardizing,

$$(*) \qquad \frac{\sum_{i=1}^{v} c_i \overline{Y}_{i\cdot} - \sum_{i=1}^{v} c_i \tau_i}{\sigma \sqrt{\sum_{i=1}^{v} \frac{c_i^2}{r_i}}} \sim \mathrm{N}(0,1)$$

Using the estimate msE for σ^2, we obtain the *standard error* for the contrast estimator $\sum_{i=1}^{v} c_i \overline{Y}_{i\cdot}$:

$$\mathrm{se}\left(\sum_{i=1}^{v} c_i \overline{Y}_{i\cdot}\right) = \sqrt{msE \sum_{i=1}^{v} \frac{c_i^2}{r_i}}\,.$$

In terms of random variables:

$$\mathrm{SE}\left(\sum_{i=1}^{v} c_i \overline{Y}_{i\cdot}\right) = \sqrt{MSE \sum_{i=1}^{v} \frac{c_i^2}{r_i}}\,.$$

Replacing the standard deviation of the contrast by the standard error in the above expression (*) gives

$$\frac{\sum_{i=1}^{v} c_i \overline{Y}_{i\cdot} - \sum_{i=1}^{v} c_i \tau_i}{\sqrt{MSE \sum_{i=1}^{v} \frac{c_i^2}{r_i}}},$$

which no longer has a normal distribution because of the substitution of msE for σ. However, the usual trick works:

$$(**) \qquad \frac{\sum_{i=1}^{v} c_i \overline{Y}_{i\cdot} - \sum_{i=1}^{v} c_i \tau_i}{\sqrt{MSE \sum_{i=1}^{v} \frac{c_i^2}{r_i}}} = \frac{\sum_{i=1}^{v} c_i \overline{Y}_{i\cdot} - \sum_{i=1}^{v} c_i \tau_i}{\sigma \sqrt{\sum_{i=1}^{v} \frac{c_i^2}{r_i}}} \Bigg/ \sqrt{MSE/\sigma^2}$$

As mentioned before, MSE/ σ^2 = SSE/(n-v) $\sigma^2 \sim \chi^2$(n-v)/(n-v). Also, it can be proved that the numerator and denominator in (**) are independent. Thus

$$\frac{\sum_{i=1}^{v} c_i \bar{Y}_{i\cdot} - \sum_{i=1}^{v} c_i \tau_i}{\sqrt{MSE \sum_{i=1}^{v} \frac{c_i^2}{r_i}}} \sim t(n-v).$$

We can now do inference for the contrast, using this test statistic.

Example: In the battery experiment, treatments 1 and 2 were alkaline batteries, while types 3 and 4 were heavy duty. To compare the alkaline with the heavy duty, we consider the difference of means contrast D = $(1/2)(\tau_1 + \tau_2) - (1/2)(\tau_3 + \tau_4)$. Find a 95% confidence interval for the contrast and perform a hypothesis test with null hypothesis: The means for the two types are equal. State precisely what the resulting confidence interval means.

Comments:

1. For a two-sided test, we could also do an F-test with test statistic t^2.

2. A very similar analysis shows that the standard error for the i^{th} treatment mean $\mu + \tau_i$ is $\sqrt{\frac{msE}{r_i}}$, and that the test statistic

$$\frac{\bar{Y}_{i\cdot} - \mu_i}{\sqrt{\frac{MSE}{r_i}}}$$

has a t-distribution with n - v degrees of freedom. This allows one to do hypothesis tests and form confidence intervals for a single mean.

3. We haven't done examples of finding confidence intervals or hypothesis tests for effect differences or for treatment means, since in practice in ANOVA, one does not do just one test or confidence interval, so modified techniques for *multiple comparisons* are needed.

The Problem of Multiple Comparisons

Suppose we want to form confidence intervals for two means or for two effect differences. If we formed a 95% confidence interval for, say, $\tau_1 - \tau_2$, and another 95% confidence interval for $\tau_3 - \tau_4$, we would get two intervals, say (a,b) and (c,d), respectively. These would mean:

1. We have produced (a,b) by a method which, for 95% percent of all completely randomized samples with the specified number in each treatment, yields an interval containing $\tau_1 - \tau_2$, and
2. We have produced (c,d) by a method which, for 95% percent of all completely randomized samples with the specified number in each treatment, yields an interval containing $\tau_3 - \tau_4$.

But there is absolutely no reason to believe that the 95% of samples in (1) are the same as the 95% of samples in (2). If we let A be the event that the confidence interval for $\tau_1 - \tau_2$ actually contains $\tau_1 - \tau_2$, and let B be the event that the confidence interval for $\tau_3 - \tau_4$ actually contains $\tau_3 - \tau_4$, the best we can say in general is the following:

P(obtaining a sample giving a confidence interval for $\tau_1 - \tau_2$ that actually contains $\tau_1 - \tau_2$ and also giving a confidence interval for $\tau_3 - \tau_4$ that actually contains $\tau_3 - \tau_4$.)

$$
\begin{aligned}
&= P(A \cap B) = 1 - P((A \cap B)^C) \\
&= 1 - P(A^C \cup B^C) \\
&= 1 - [P(A^C) + P(B^C) - P(A^C \cap B^C)] \\
&= 1 - P(A^C) - P(B^C) + P(A^C \cap B^C) \\
&\geq 1 - P(A^C) - P(B^C) = 1 - 0.05 - 0.05 = 0.90
\end{aligned}
$$

Similarly, if we were forming m 95% confidence intervals, our "confidence" that for all of them, the corresponding true effect difference would lie in the corresponding CI would, by this reasoning, be reduced to 1 - .05m.

Thus, other techniques are needed for such "simultaneous confidence intervals" or "multiple comparisons." (Similar comments apply to simultaneous hypothesis tests.)