# TWO WAY ANALYSIS OF VARIANCE MODEL

Consider a completely randomized design for an experiment with two treatment factors A and B.

Assume that the factors are ***crossed***: This means that *every level of A is observed with every level of B*.

*Notation*:

A has a levels coded 1, 2, … , a

B has b levels coded 1,2, … , b

v = total number of treatments ( = ab)

*Example*: In the battery example, we had two factors:

A with levels 1 = alkaline, 2 = heavy duty

B with levels 1 = name brand, 2 = store brand

So a = 2, b = 2, v = 4.



When designing the experiment, we could imagine various scenarios. Four of many possibilities:

I) The means of LPUC for levels of B:

- Do not depend on the level of A, and
- Are higher for level 1 of B than for level 2 of B.

Picturing this with an *interaction plot*:

```
                        Brand: Name = 1, Store = 2
         |
         |
Group    |   1--------1
 mean    |
         |   2--------2
         |
         |_____________________
             1        2
         Type battery: Alkaline = 1, Heavy duty = 2
```

We could also sketch an interaction plot with the roles of the factors reversed:

Group mean

Type battery:
Alkaline = 1,
Heavy duty = 2

1 2

Brand: Name = 1, Store = 2



II) Treatment makes no difference in LPUC.

Brand: Name = 1, Store = 2

Group mean

1 2

Type battery: Alkaline = 1, Heavy duty = 2

Group mean

Type battery:
Alkaline = 1,
Heavy duty = 2

1 2

Brand: Name = 1, Store = 2

III) The mean levels for LPUC are:

- Higher for level 1 of B than for level 2 of B when A is at level 1
- The same for both levels of B when A is at level 2

Brand: Name = 1, Store = 2

Group mean

1 2

Type battery: Alkaline = 1, Heavy duty = 2

Group mean

Type battery:
Alkaline = 1,
Heavy duty = 2

1 2

Brand: Name = 1, Store = 2



IV) The mean LPUC is

- Higher for level 1 of B than for level 2 of B when A is at level 1
- Lower for level 1 of B than for level 2 of B when A is at level 2.

Brand: Name = 1, Store = 2

Group mean

1 2

Type battery: Alkaline = 1, Heavy duty = 2

Group mean

Type battery:
Alkaline = 1,
Heavy duty = 2

1 2

Brand: Name = 1, Store = 2

(We can also form interaction plots when factors have more than two levels; see p. 137 for examples.)

**Possible Models for Two Crossed Factors**

Let $Y_{ijt}$ denote the random variable giving the response for observation t of the treatment at level i of A and level j of B.

($r_{ij}$ = number of observations at level i of A and level j of B.)

1. The *cell-means model*:

$$Y_{ijt} = \mu + \tau_{ij} + \varepsilon_{ijt}$$

The $\varepsilon_{ijt}$ are independent random variables.

Each $\varepsilon_{ijt} \sim N(0, \sigma^2)$.

*Exercise*: What can you say about the $\tau_{ij}$'s in each scenario (I) - (IV) above when using the cell-means model?



2. The *main effects model* (also known as the *two-way additive model*):

$$Y_{ijt} = \mu + \alpha_i + \beta_j + \varepsilon_{ijt}$$

The $\varepsilon_{ijt}$ are independent random variables

Each $\varepsilon_{ijt} \sim N(0, \sigma^2)$

*Exercise*:

a. Can each of scenarios I - IV be modeled by the main effects model?

If not, which ones can and which ones can't?

In any of the scenarios that can be modeled by the main effect model, what is the connection between $\alpha_i$, $\beta_j$ of this model and $\tau_{ij}$ of the cell-means model?

b. For a 2 x 2 crossed design (i.e., two crossed factors – e.g., the battery situation), find $E(Y_{11} - Y_{12})$ and $E(Y_{21} - Y_{22})$ in the main effects model. What are the implications of what you find?

3. The *two-way analysis of variance model* (also known as the *two-way complete model*):

$$Y_{ijt} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijt}$$

The $\varepsilon_{ijt}$ are independent random variables.

Each $\varepsilon_{ijt} \sim N(0, \sigma^2)$

Note that:

- The two-way ANOVA model is equivalent to the cell-means model: $\tau_{ij} = \alpha_i + \beta_j + (\alpha\beta)_{ij}$
- The main effects model is a submodel of the two-way ANOVA model -- the special case of the two-way model when all $(\alpha\beta)_{ij} = 0$.

The terms $(\alpha\beta)_{ij}$ are called *interaction terms*. The following exercise illustrates why.

*Exercise*: In each scenario I - IV above, give values of the parameters $\alpha_i$, $\beta_j$, and $(\alpha\beta)_{ij}$ that are consistent with the scenario.