ANALYSIS OF BALANCED FACTORIAL DESIGNS

(Discussion applies to the Complete Model with any number of factors, but will be illustrated with the Three-Way Complete Model.)

We can use Least Squares to get estimates of model parameters and contrasts. (Additional constraints must be added to estimate non-estimable parameters.)

Example: The cell means are estimable. For three factors, the Least Squares estimates of the cell means are $\bar{y}_{ijk\bullet}$.

From the cell-means model and one-way analysis of variance, we have:

- The *error sum of squares* ssE is the sum of squared deviations from the fits $\bar{y}_{ijk\bullet}$.

  Example: For 3 factors, ssE = $\sum_i \sum_j \sum_k \sum_t (y_{ijkt} - \bar{y}_{ijk\bullet})^2$

- ssE has associated degrees of freedom n – v.

  Example: For 3 factors (balanced design), ssE has abcr – abc = abc(r-1) degrees of freedom.

- The *mean square error* msE is ssE divided by its degrees of freedom.

  Example: For 3 factors, msE = ssE/ abc(r-1).

- msE is an estimate for the variance σ^2.

- The standard error of the residuals $y_{ijkt} - \bar{y}_{ijk\bullet}$ is $\sqrt{\dfrac{ssE}{n-1}}$

Before doing inference,

1. Decide which tests/comparisons we will do.

2. Use fits and residuals to check the model assumptions of independent, normal errors with constant variance.

Example: Pollution filter data

1. Decide which tests/comparisons we will do and how type I error will be split up.

2. Note: To get a column for treatment easily, use

100*SIZE + 10*TYPE + SIDE

Hypothesis Tests and Analysis of Variance Table (3-Way Models, Balanced Design)**:**

Submodels can be tested against larger models by F-tests, with F-statistic obtained as a ratio of mean squares.

The *error sum of squares* of a model is the sum of the squared deviations from the fits from that model.

Example: For three factors, the error sum of squares for a model is

$$\sum_i \sum_j \sum_k \sum_t (y_{ijkt} - \hat{y})^2,$$

where $\hat{y}$ is the fit for the model.

Special case I: Since the fit for the complete model is $\bar{y}_{ijk\bullet}$, the error sum of squares for the complete model is ssE mentioned above.

Special case II: The *total sum of squares* is the error sum of squares for the model

$$y_{ijkt} = \mu + \varepsilon_{ijkt}$$

Since the fit for this model is the overall sample mean, the total sum of squares can be described as the sum of the squared deviations from the overall sample mean.

Example: For 3 factors,

$$\text{sstot} = \sum_i \sum_j \sum_k \sum_t (y_{ijkt} - \bar{y}_{....})^2$$

The total sum of squares has n-1 associated degrees of freedom, where n = total number of observations (e.g., abcr for 3 factors).

Recall that for three factors, the null hypothesis for "no three-way interaction" is

H_0^{ABC}:

$$[(\alpha\beta\gamma)_{i+1,jk} - (\alpha\beta\gamma)_{ijk}] - [(\alpha\beta\gamma)_{i+1,qk} - (\alpha\beta\gamma)_{iqk}]$$

$$- \; [(\alpha\beta\gamma)_{i+1,jr} - (\alpha\beta\gamma)_{ijr}] - [(\alpha\beta\gamma)_{i+1,qr} - (\alpha\beta\gamma)_{iqr}]$$

$$= 0 \text{ for all i, j, k, q, r}$$

If we let ssE_0^{ABC} denote the error sum of squares for the submodel where H_0^{ABC} is true, then we define *the sum of squares for three-way interaction* to be

$$ssABC = ssE_0^{ABC} - ssE$$

It has associated degrees of freedom (a-1)(b-1)(c-1).

Define the *mean square error for ABC* as

$$msABC = ssABC/ \, (a-1)(b-1)(c-1).$$

Then MSABC/MSE has an F distribution with (a-1)(b-1)(c-1) and n-v degrees of freedom. This gives us an F-test for H_0^{ABC}.

Example: Pollution filter data.

Tests for Two-Way Interactions

Analogously to the test for main effects in 2-way ANOVA, the tests for 2-way interaction in 3-way ANOVA have hypotheses:

H_0^{BC}: All $(\beta\gamma)_{ij}$*'s are equal

H_a^{BC}: At least two of the $(\beta\gamma)_{ij}$*'s are different,

where the $(\beta\gamma)_{ij}$*'s are defined in a manner analogous to the α_i*'s for two-way ANOVA, as the $(\beta\gamma)_{ij}$'s plus averages over the higher interaction terms.

Thus the test is whether or not the "levels" of BC interaction, *averaged over the levels of the other factors*, have the same average effect on the response.

Thus define

ssE_0^{BC} = the sum of squares for the model assuming H_0^{BC} is true,

and *the sum of squares for BC*

$ssBC = ssE_0^{BC} - ssE$.

Then ssBC has (b-1)(c-1) degrees of freedom.

Defining msBC = ssBC/(b-1)(c-1), we get F-statistic MSBC/MSE, with degrees of freedom (b-1)(c-1) and n-v. This gives us an F-test for H_0^{BC}.

We proceed analogously to obtain hypothesis tests for the other two-way interactions and for "main effects" of A, B, and C.

In general (for *balanced* designs with *complete* model), the degrees of freedom for a sum of squares for an item in the ANOVA table is the product of one less than the number of levels for each factor involved in the item.

Examples:

Sum of squares for	Degrees of freedom
A	a-1
AB	(a-1)(b-1)
BC	(b-1)(c-1)
ABC	(a-1)(b-1)(c-1)

The total sum of squares partitions into (is the sum of) the error sum of squares plus the sums of squares for all items in the model.

The degrees of freedom add correspondingly. For example, for the complete 3-way model, this is just the algebraic identity

$$(a-1)(b-1)(c-1) + (a-1)(b-1) + (a-1)(c-1) + (b-1)(c-1)$$

$$+ (a-1) + (b-1) + (c-1) + abcr - abc$$

$$= abcr - 1$$

Note: If we are dealing with a model other than the complete model, the error sum of squares for that model will have a different number of degrees of freedom. The degrees of freedom for error are usually most easily found by subtracting the other degrees of freedom from the total degrees of freedom.

The *mean square* for an item is the sum of squares for that item divided by its degrees of freedom.

Example: msBC = ssBC/(b-1)(c-1)

The F-statistic for an item is the mean square for the item divided by the mean square error, with corresponding degrees of freedom in the numerator in denominator.

Contrasts

Contrasts are defined and estimated as in the cell means model.

Various labels that are given to certain types of contrasts; see p. 199 for details.

Contrasts can also be defined relative to submodels, in which case parameters not present in the submodel are omitted from the contrast.

The various methods for simultaneous confidence intervals still apply.

There is a modification that can be used for finding simultaneous confidence intervals for contrasts in the levels of a single factor: Replace v by the number of levels of the factor in question, and replace r by the number of observations on each level of the factor of interest. (See p. 205 for an example.)

Example: Continuing with the Pollution Filter Data, the main interest is in comparing levels of Type (the second factor).

Comments:

- We can also do *unbalanced* ANOVA via GLM, similarly to the 2-way case.
- Similar considerations/variations hold for "small experiments" (i.e., one observation per cell.)
- Sometimes transformations are helpful (e.g., to obtain constant variance or normality, or to remove interaction).
- Other approaches:

    Non-parametric
        e.g., Kruskall-Wallis:
            χ^2 test based on rankings
            Doesn't require normality
            Less powerful than ANOVA

    Bayesian methods