INFERENCE FOR ONE-WAY ANOVA

To test equality of means for different treatments/levels:

H_0: $\mu_1 = \mu_2 = \ldots = \mu_v$

Rephrase:

- In terms of effects:

- In terms of differences of effects:

- In terms of contrasts $\tau_i - \bar{\tau}$, where $\bar{\tau} = \frac{1}{v}\sum_{i=1}^{v}\tau_i$:

Treatment degrees of freedom = minimum number of equations needed to state the null hypothesis = ___

Alternate hypothesis:

H_a:

Idea of the test:

Compare:

ssE under the *full* model (with all parameters)

and

ssE_0 -- the error sum of squares under the *reduced* model -- i.e., the one assuming H_0 is true.

To calculate ssE_0:

If H_0 is true, let τ be the common value of the τ_i's. Then the reduced model is:

- $Y_{it} = \mu + \tau + \varepsilon_{it}^0$
- $\varepsilon_{it}^0 \sim N(0, \sigma^2)$
- the ε_{it}^0's are independent,

where ε_{it}^0 denotes the it[th] error in the *reduced* model.

To find ssE_0: Use least squares to minimize

$$g(m) = \sum_{i=1}^{v}\sum_{t=1}^{r_i}(y_{it} - m)^2:$$

$$g'(m) = \sum_{i=1}^{v}\sum_{t=1}^{r_i}2(-1)(y_{it} - m) = 0,$$

which yields the estimate $\bar{y}_{\cdot\cdot}$ for $\mu + \tau$.

i.e., the least squares estimate of $\mu + \tau$ is

$$(\mu + \tau)^\wedge = \bar{y}_{\cdot\cdot}\ .$$

(By abuse of notation, called $\hat{\mu} + \hat{\tau}$).

So

$$ssE_0 = \sum_{i=1}^{v}\sum_{t=1}^{r_i}(y_{it} - \bar{y}_{\cdot\cdot})^2,$$

which can be shown (proof might be homework) to equal

$$\sum_{i=1}^{v}\sum_{t=1}^{r_i}y_{it}^2 - n(\bar{y}_{\cdot\cdot})^2$$

Note that ssE and ssE_0 can be considered as minimizing the same expression, but over different sets:

ssE minimizes $\sum_{i=1}^{t}\sum_{t=1}^{r_i}(y_{it} - m - t_i)^2$ over the set of all v + 1-tuples $(m, t_1, t_2, \ldots, t_v)$

ssE_0 can be considered as minimizing the same expression over the subset where all t_i's are 0. Therefore ssE_0 must be at least as large as ssE:

$$ssE_0 \geq ssE.$$

If H_0 is true, ssE and ssE_0 should be about the same.

This suggests: Use the ratio $(ssE_0\text{-}ssE)/ssE$ as a test statistic for the null hypothesis:

If H_0 is true, this ratio should be small, so an unusually large ratio would be reason to reject the null hypothesis.

The difference ssE_0-ssE is called the *sum of squares for treatment,* or *treatment sum of squares*, denoted ssT.

Using the alternate expressions for ssE_0 and ssE:

$$ssT = ssE_0 - ssE = \sum_{i=1}^{v}\sum_{t=1}^{r_i} y_{it}^2 - n(\bar{y}..)^2 - \left(\sum_{i=1}^{v}\sum_{t=1}^{r_i} y_{it}^2 - \sum_{i=1}^{v} r_i(\bar{y}_{i\bullet})^2\right)$$

$$= \sum_{i=1}^{v} r_i(\bar{y}_{i\bullet})^2 - n(\bar{y}..)^2$$

$$= \sum_{i=1}^{v} \frac{(y_{i\bullet})^2}{r_i} - \frac{(y..)^2}{n} \quad \text{(using definitions)}$$

$$= \sum_{i=1}^{v} r_i(\bar{y}_{i\bullet} - \bar{y}..)^2 \quad \text{(possible homework)}$$

This last expression can be considered as a "between treatments" sum of squares --- we are comparing each treatment sample mean $\bar{y}_{i\bullet}$ with the grand (overall) mean $\bar{y}..$

By contrast, our denominator, $ssE = \sum_{i=1}^{v}\sum_{t=1}^{r_i}(y_{it} - \bar{y}_i)^2$ is a "within treatments" sum of squares: it compares each value with the mean for the treatment group from which the value was obtained.

Using the model assumptions, it can be proved that:

- $ssE/\sigma^2 \sim \chi^2(n - v)$
- If H_0 is true, $ssT/\sigma^2 \sim \chi^2(v - 1)$
- If H_0 is true, then ssT and ssE are independent.

Thus, <u>if</u> H_0 is true

$$\frac{ssT/\sigma^2(v-1)}{ssE/\sigma^2(n-v)} \sim F_{v-1,n-v}.$$

Now $\frac{ssT/\sigma^2(v-1)}{ssE/\sigma^2(n-v)}$ simplifies to $\frac{ssT/(v-1)}{ssE/(n-v)}$, which we can calculate from our sample.

We originally wanted to test ssT/ssE, but $\frac{ssT/(v-1)}{ssE/(n-v)}$ is just a constant multiple of ssT/ssE, so good enough for our purposes:

$\frac{ssT/(v-1)}{ssE/(n-v)}$ is unusually large exactly when ssT/ssE is unusually large.

Thus, we can use an F test, with test statistic

$\frac{ssT/(v-1)}{ssE/(n-v)}$, to test our hypothesis.

Note: We can look at ssT/(v-1) and ssE/(n-v) as we did in the equal-variance, two-sample t-test:

- ssE/(n-v) is a pooled estimate of the common variance σ^2
- If H_0 is true, then ssT/(v - 1) can be regarded as another estimate of σ^2.

Notation:

ssT/(v-1) is called msT (*mean square for treatment* or *treatment mean square*

ssE/(n-v) is called msE (*mean square for error* or *error mean square*).

So the test statistic is F = msT/msE.