CONFIDENCE BANDS

Under the simple linear model, we obtained the standard error of $\hat{\mathrm{E}}(\mathrm{Y}|\mathrm{x})$:

$$\text{s.e}\,(\hat{\mathrm{E}}(\mathrm{Y}|\mathrm{x})) = \hat{\sigma}\sqrt{\frac{1}{n}+\frac{(x-\bar{x})^2}{SXX}}$$

If we plot the least squares regression line, and then for each point (x,y) on the line plot the points (x,y ± s.e $(\hat{\mathrm{E}}(\mathrm{Y}|\mathrm{x}))$, we will get two curves, with equations

$$y = \hat{\eta}_0 + \hat{\eta}_1 x \pm \hat{\sigma}\sqrt{\frac{1}{n}+\frac{(x-\bar{x})^2}{SXX}}.$$

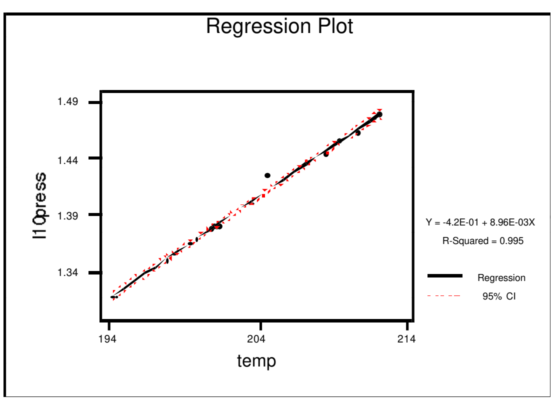
What kinds of curves are these? We will answer this a little more generally, looking at curves of the form

$$(*) \qquad y = \hat{\eta}_0 + \hat{\eta}_1 x \pm c\sqrt{\frac{1}{n}+\frac{(x-\bar{x})^2}{SXX}},$$

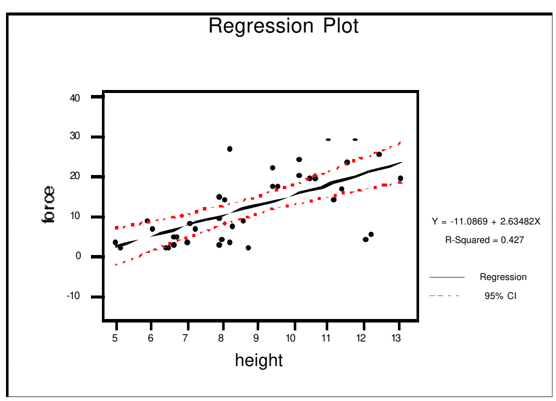
for some constant c. These are called *confidence bands*. For example, if we choose c = $t\hat{\sigma}$, where t is the 95th percentile for the t(n-2) distribution, then the curves will show the 90% confidence intervals for $\hat{\mathrm{E}}(\mathrm{Y}|\mathrm{x})$ as x varies.

Example of confidence bands from Minitab:

Forbes data



Another example



We need the following criterion for determining what type of curve a quadratic equation in x and y describes:

Given the quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

if the *discriminant* $B^2 - 4AC$ is positive, then the graph of the equation is a hyporbola (or a pair of intersecting lines in the degenerate case). (For more information, see the Mathworld website at http://mathworld.wolfram.com/QuadraticCurveDiscriminant.html)

A little algebraic manipulation puts our equation (*) in the form

$$(y - \hat{\eta}_0 - \hat{\eta}_1 x)^2 = c^2\left(\frac{1}{n} + \frac{(x-\bar{x})^2}{SXX}\right).$$

More algebra gives

$$y^2 - 2\hat{\eta}_1 xy + \hat{\eta}_1^2 x^2 - \frac{c^2}{SXX}x^2 + \text{(terms of degree 1 and 2)} = 0.$$

So $A = \hat{\eta}_1^2 - \frac{c^2}{SXX}$, $B = -2\hat{\eta}_1$, and $C = 1$, giving

$$B^2 - 4AC = 4\hat{\eta}_1^2 - 4[\hat{\eta}_1^2 - \frac{c^2}{SXX}] = \frac{c^2}{SXX} > 0,$$

so the confidence bands are a hyperbola.

[Note: The least squares regression line is *not* one of the axes of the hyperbola, since the confidence bands are "equidistant " from the line vertically, but not in the perpendicular direction.]