INDEPENDENCE, COVARIANCE AND CORRELATION M384G/374G

Independence: For random variables X and Y, the intuitive idea behind "Y is independent of X" is that the distribution of Y shouldn't depend on what X is. This can be expressed in terms of the conditional pdf's to say "f(y|x) doesn't depend on x."

Caution: "Y is not independent of X" means simply that the *distribution* of X may vary as Y varies. It *doesn't* mean that X is a function of Y.

If Y is independent of X, then:

1. $\mu_x = E(Y|X = x)$ does not depend on x.

(*Question*: Is the converse true? That is, if E(Y|X = x) does not depend on x, can we conclude that Y is independent of X?)

2. (Still assuming Y is independent of X) Let h(y) be the common pdf of the conditional distributions Y|X. Then for every x, $h(y) = f(y|x) = \dfrac{f(x,y)}{f_X(x)}$, where f(x,y) is the joint pdf of X and Y. Therefore

$$f(x,y) = h(y)\, f_X(x)$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \\ &= \int_{-\infty}^{\infty} h(y)\, f_X(x)\,dx \\ &= h(y)\int_{-\infty}^{\infty} f_X(x)\,dx = h(y) = f(y|x) \end{aligned}$$

In other words: *If Y is independent of X, then the conditional distributions of Y given X are the same as the marginal distribution of Y.*

3. Now (still assuming Y is independent of X) we have

$$f_Y(y) = f(y|x) = \frac{f(x,y)}{f_X(x)},$$

so

$$f_Y(y) f_X(x) = f(x,y).$$

In other words: *If Y is independent of X, then the joint distribution of X and Y is the product of the marginal distributions of X and Y.*

<u>Exercise</u>: The converse of this last statement is true. That is: If the joint distribution of X and Y is the product of the marginal distributions of X and Y, then Y is independent of X.

Note that the condition $f_Y(y)f_X(x) = f(x,y)$ is symmetric in X and Y. Thus (3) and its converse imply that : Y is independent of X if and only if X is independent of Y. So it makes sense to say "X and Y are independent."

Putting this all together, have: The following conditions are all equivalent:

i. X and Y are independent.
ii. $f_{X,Y}(x,y) = f_Y(y)f_X(x)$
iii. The conditional distribution of $Y|X = x$ is independent of x
iv. The conditional distribution of $X|Y = y$ is independent of y.
v. $f(y|x) = f_Y(y)$ for all y.
vi. $f(x|y) = f_X(x)$ for all x.

Additional property of independent random variables: If X and Y are independent, then E(XY) = E(X)E(Y). (*The proof of this fact will be assigned as homework for October 14.)*

Covariance: The *covariance* of two random variables X and Y is defined as

$$\text{Cov}(X,Y) = E([X - E(X)][Y - E(Y)])$$

Comments:

- The capital C in Cov is consistent with the notation used in this class of capitalizing items that relate to the population, and using lower case (or a "hat") for items referring to a sample. There is a related notion of covariance for a sample, discussed briefly later. Consistent with general terminology, Cov is a *parameter* since it refers to the population, and the sample covariance (cov or Cov-hat) is a *statistic* since it is calculated from the sample.
- Compare and contrast with the definition of Var(X).
- If X and Y both tend to be on the same side of their respective means (i.e., both greater than or both less than their means), then [X - E(X)][Y - E(Y)] tends to be positive, so Cov(X,Y) is positive. Similarly, if X and Y tend to be on opposite sides of their respective means, then Cov(X,Y) is negative. If there is no trend of either sort, then Cov(X,Y) should be zero. Thus covariance roughly measures the extent of a "positive" or "negative" trend in the joint distribution of X and Y.
- What are the units of Cov(X,Y)?

***Properties*:**

1. Cov(X, X) =

2. Cov(Y, X) =

3. Cov (X, Y) = E(XY) - E(X)E(Y).
   - Why?
   - In words …

- Compare with the alternate formula for Var(X).

4. Consequence: If X and Y are independent, then:

   Note: The converse of this statement is false. This will be a problem on a future homework set.

5. Cov(cX, Y) = and Cov(X, cY) =

6. Cov(a + X,Y) = and Cov(X, a +Y) =

7. Cov(X + Y, Z)) = and Cov(X, Y + Z) =

8. Var(X + Y) = Var(X) + 2Cov(X, Y) + Var(Y)
   - Why?
   - Consequence: If X and Y are independent, then ____________________________
   - *Note*: The converse of this last statement is false.

Bounds on Covariance

Let σ_X denote the *population standard deviation* $\sqrt{Var(X)}$ of X. (Do not confuse with the sample standard deviation = s or s.d. or $\hat{\sigma}$). Define the population standard deviation σ_Y of Y similarly.

Consider the new random variable $\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}$. Since Variance is always ≥ 0,

(*) $$0 \le \text{Var}\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right)$$

$$= \text{Var}\left(\frac{X}{\sigma_X}\right) + \text{Var}\left(\frac{Y}{\sigma_Y}\right) + 2\text{Cov}\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y}\right)$$

$$= \frac{1}{\sigma_x^2}\text{Var(X)} + \frac{1}{\sigma_Y^2}\text{Var(Y)} + \frac{2}{\sigma_X\sigma_Y}\text{Cov(X,Y)}$$

$$= 2\left[1 + \frac{Cov(X,Y)}{\sigma_X\sigma_Y}\right].$$

Therefore

(**) $$\frac{Cov(X,Y)}{\sigma_X\sigma_Y} \ge -1$$ (or: Cov(X, Y) ≥ $-\sigma_X\sigma_Y$).

Looking at $\text{Var}\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right)$ similarly shows (details left to the student):

(***) $$\frac{Cov(X,Y)}{\sigma_X\sigma_Y} \leq 1,$$ (or: Cov(X, Y) ≤ $\sigma_X\sigma_Y$.).

Combining (**) and (***) gives:

$$\left|\frac{Cov(X,Y)}{\sigma_X\sigma_Y}\right| \leq 1,$$ (or: |Cov(X, Y)| ≤ $\sigma_X\sigma_Y$)

Moreover, the only way we can have equality in inequality (**) is to have equality in (*) -- i.e, when

$$\text{Var}\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) = 0$$

This can happen if and only if the random variable $\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}$ is constant -- say,

$$\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y} = \text{c}.$$

(Note that c must be the mean of $\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}$, which is $\frac{\mu_X}{\sigma_X} + \frac{\mu_Y}{\sigma_Y}$)

This in turn is equivalent to

$$Y = \sigma_Y\left(-\frac{X}{\sigma_X} + c\right)$$

or

$$Y = -\frac{\sigma_Y}{\sigma_X}X + \sigma_Y c,$$

which says: The pairs (X,Y) lie on a line with negative slope.(The converse is also true -- details left to the student. Also note that the slope of the line is $-\frac{\sigma_Y}{\sigma_X}$ and the y-intercept is $\frac{\sigma_Y}{\sigma_X}\mu_X + \mu_Y$.)

Similarly, $\frac{Cov(X,Y)}{\sigma_X\sigma_Y}$ = +1 exactly when the pairs (X,Y) lie on a line with *positive* slope.

Correlation: The *correlation coefficient* of the random variables X and Y is

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}.$$

Note:

- $\rho_{X,Y}$ is often called ρ for short.
- ρ is a parameter (since it refers to the population). There is also a sample correlation coefficient (usually called r) that is a statistic (calculated from the sample).

Stated in terms of ρ, the discussion above says:

- Negative ρ indicates a tendency for the variables X and Y to co-vary in a negative way.
- Positive ρ indicates a tendency for the variables X and Y to co-vary in a positive way.
- $-1 \le \rho \le 1$
- $\rho = -1$ if and only if all pairs (X,Y) lie on a straight line with negative slope.
- $\rho = 1$ if and only if all pairs (X,Y) lie on a straight line with positive slope.
- ρ is unitless.
- ρ is the Covariance of the standardized random variables $\frac{X-\mu_X}{\sigma_X}$ and $\frac{Y-\mu_Y}{\sigma_Y}$.

  (Details left to the student.)

Also, from the definition, we see that $\rho = 0$ if and only if Cov(X,Y) = 0.

Uncorrelated variables: We say that two random variables are *uncorrelated* if $\rho_{X,Y} = 0$ (or equivalently, if Cov(X,Y) = 0).

Examples:

- If X and Y are independent, then they are uncorrelated. (Why?)
- Suppose that the random variable X is uniform on the interval [-1, 1]. Let $Y = X^2$. Then X and Y are uncorrelated, but not independent. (To see that X and Y are not independent, note that E(Y|X) is not constant. Details of why X and Y are uncorrelated will be on the next homework assignment.)

In general, ρ is a measure of the degree of a *nonconstant linear* relationship between X and Y. Example 2 above shows that two variables can have a strong nonlinear relationship and still be uncorrelated.

Sample variance, covariance, and correlation

If we have a sample of data (x_1,y_1), (x_2,y_2), … , (x_n,y_n) from the joint distribution of X and Y, we can define the statistics

sample covariance cov(x,y) (or Cov-hat(x,y)) = $\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})$

and

sample correlation coefficient r (or $\hat{\rho}$) = $\frac{\text{cov}(x,y)}{sd(x)sd(y)}$.

These are estimators of the corresponding population parameters. We can establish properties of the sample covariance and correlation coefficient analogous to those of the population covariance and correlation coefficient.