INFERENCE FOR SIMPLE OLS

Model Assumptions *("The" Simple Linear Regression Model Version IV):*
(We consider $x_1, \dots, x_n$ as fixed.)

1. $E(Y|x) = \eta_0 + \eta_1 x$ (linear mean function)
2. $Var(Y|x) = \sigma^2$ (Equivalently, $Var(e|x) = \sigma^2$) (constant variance)
3. $y_1, \dots, y_n$ are independent observations. (independence)
4. (*NEW)* Y|x is normal for each x (normality)

(1) + (2) + (4) can be summarized as:

$$Y|x \sim N(\eta_0 + \eta_1 x, \sigma^2)$$

Recall: $e|x = Y|x - E(Y|x)$

So: $e|x \sim N(0, \sigma^2)$

i.e., all errors have the same distribution -- so we just say e instead of e|x .

Since $\hat{\eta}_0$ and $\hat{\eta}_1$ are linear combinations of the $Y|x_i$'s, (3) + (4) imply that $\hat{\eta}_0$ and $\hat{\eta}_1$ are normally distributed random variables (that is, their sampling distributions are normal). Recalling that

$$E(\hat{\eta}_1) = \eta_1 \qquad Var((\hat{\eta}_1) = \frac{\sigma^2}{SXX} \qquad E(\hat{\eta}_0) = \eta_0 \qquad Var\,(\hat{\eta}_0) = \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{SXX}\right),$$

We have

$$\hat{\eta}_1 \sim \qquad\qquad\qquad \hat{\eta}_0 \sim$$

Look more at $\hat{\eta}_1$: We can standardize to get

$$\frac{\hat{\eta}_1 - \eta_1}{\sqrt{\sigma^2 / SXX}} \sim N(0,1)$$

But we don't know σ^2, so need to approximate it by $\hat{\sigma}^2$ -- in other words approximate $Var(\hat{\eta}_1)$ by $V\hat{a}r(\hat{\eta}_1)$= [s.e. $(\hat{\eta}_1)]^2 = \frac{\hat{\sigma}^2}{SXX}$. Thus we want to use $\frac{\hat{\eta}_1 - \eta_1}{\sqrt{\hat{\sigma}^2 / SXX}}$. But we can't expect this to be normal, too. However,

$$\frac{\hat{\eta}_1 - \eta_1}{\sqrt{\hat{\sigma}^2 / SXX}} =$$

(*) $$\frac{\dfrac{\hat{\eta}_1 - \eta_1}{\sqrt{\sigma^2 / SXX}}}{\sqrt{\hat{\sigma}^2 / \sigma^2}}$$

The numerator of the last fraction *is* normal (in fact, standard normal), as noted above.

Facts: (Proofs omitted)

a. (n-2)$\dfrac{\hat{\sigma}^2}{\sigma^2}$ has a χ^2 distribution with n-2 degrees of freedom

Notation: (n-2)$\dfrac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2$(n-2)

b. (n-2)$\dfrac{\hat{\sigma}^2}{\sigma^2}$ is independent of $\hat{\eta}_1 - \eta_1$ (hence independent of the numerator in (*))

Comments on distributions:

1. A χ^2(k) distribution is defined as the distribution of a random variable which is a sum of squares of k independent standard normal random variables.

[Comment: Recall that $\hat{\sigma}^2 = \dfrac{1}{n-2} RSS$, so (n-2)$\dfrac{\hat{\sigma}^2}{\sigma^2} = \dfrac{RSS}{\sigma^2} = \sum \left(\dfrac{\hat{e}_i}{\sigma}\right)^2$ *is* a sum of n squares; the fact quoted above says that it can also be expressed as a sum of n-2 squares of *independent standard normal* random variables.]

2. A t-distribution with k degrees of freedom is defined as the distribution of a random variable of the form $\dfrac{Z}{\sqrt{U/k}}$ where

- Z~N(0,1)
- U~ χ^2(k)
- Z and U are independent.

In the fraction (*) above, take

U = (n-2) $\dfrac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2$(n-2)

Z = $\dfrac{\hat{\eta}_1 - \eta_1}{\sqrt{\sigma^2 / SXX}} \sim$ N(0,1)

Thus: $$\frac{\hat{\eta}_1 - \eta_1}{\sqrt{\hat{\sigma}^2 / SXX}} \sim t(n-2),$$

so we can do inference on η_1, using t = $\frac{\hat{\eta}_1 - \eta_1}{\sqrt{\hat{\sigma}^2 / SXX}}$ as our test statistic.

Inference on η_0

With the same assumptions, it can be shown in an analogous manner (details omitted) that

$$\frac{\hat{\eta}_0 - \eta_0}{s.e.(\hat{\eta}_0)} \sim t(n-2),$$

so we can use this statistic to do inference on η_0.