LOGARITHMS AND MEANS

I. Acidity/alkalinity is measured by the pH scale, which measures the hydrogen ion activity of a substance. In more detail:

$$pH = -\log_{10} [a_{H+}],$$

where a_{H+} is the activity of H^+ ions in the substance – that is, the "effective concentration" of H^+ ions. In most common cases, this is approximately $[H^+]$ = the concentration of the H^+ ion measured in moles per liter. (A *mole* of a substance is the amount in grams equal to its atomic weight.) Thus an approximate definition that is often used is

$$pH = -\log_{10} [H^+],$$

For most substances, pH ranges from 0 to 14 (although acid mine runoff can have a pH of -3.6).

a. Using the approximate definition, explain why

i. a drop by one pH unit represents a 10-fold increase in the concentration of hydrogen ions (making the substance more acidic) *Note*: "10-fold increase" means that the H ion concentration after the increase is <u>ten times</u> what it was before the increase.

ii. an increase by one pH unit means a 10-fold decrease in concentration of hydrogen ions (making the substance more alkaline). *Note*: "10-fold decrease" similarly means that the H ion concentration is <u>one tenth</u> what is was before the decrease.

b. i. What is $[H^+]$ for a substance of pH 7? For one of pH 2? For one of pH 13?

ii. Explain why pH is used rather than ion concentration itself. [Hint: part (i)]

c. Explain why a logarithmic scale could also be called a multiplicative (rather than the more usual additive) scale.

d. According to an article in the March, 2006 issue of Scientific American (p. 61), "The absorption of carbon dioxide has already caused the pH of modern surface waters to be about 0.1 lower (less alkaline) than it was in pre-industrial times" If the H-ion concentration of surface water in pre-industrial times was C, what is it now?

e. The article also says that "unless civilization modifies its appetite for fossil fuels soon and in a significant way, ocean pH will fall an additional 0.3 by 2100." In terms of C, what would the ocean H - ion concentration be in 2100 under these conditions?

II.

a. Suppose you start with numbers $a_1, a_2, \ldots, a_n$, take their logs (let's all use log 10 to be consistent), then take the average (arithmetic mean) of what you get, then raise 10 to that power. What do you get (as a formula in terms of $a_1, a_2, \ldots, a_n$)? This is called the *geometric mean* of $a_1, a_2, \ldots, a_n$.

b. You can think of the arithmetic mean as the number that answers the questions, "If all of the quantities had the same value, what would that value have to be to in order to get the same total?" What analogous question does the geometric mean answer?

c. Which (arithmetic or geometric mean) would be the better measure of "average" in each of the following cases? Give reasons.

i. Hydrogen ion concentrations.

ii. pH's

iii. The "average growth factor of a population". What this means is: We say "a population grows by a factor of f during a certain time period" if the population starts at size P at the beginning of the time period and ends at size P at the end of that time period. Now suppose a certain population grows by a factor of f_1 in time period 1, then by a factor of f_2 in time period 2, … , and by a factor f_n in time period n, where all the time periods are of equal length. What would make sense as "average growth factor" in this context?

d. You have an investment which earns simple interest at a rate r_1% the first year, at a rate r_2% the second year, … , and at a rate r_n% the nth year. If it grew at the same interest rate r each of the n years, what would r need to be to give the same end result? Explain how this might reasonably be considered the "average rate of return." How is this question related to the geometric mean?

e. The following quote is from p. 54 of the article "Wading in Waste", by Mallin, Michael A., *Scientific American*, 00368733, Jun2006, Vol. 294, Issue 6:

"*The U.S. Public Health Service has set a nationwide safety standard for shellfish beds using measurements of fecal coliform bacteria, a broad category of microorganisms found in the intestines of humans and animals. Shellfish cannot be harvested from the area if the geometric mean of the bacterial counts in 30 sets of samples is higher than 14 colony-forming units (CFU) per 100 milliliters of seawater. (A geometric mean is a type of average that minimizes the effects of outlying values.)* "

Outlying values are data values that are either noticeably smaller or noticeably larger than other data values occurring.

i. Try some examples to check out whether or not the geometric mean minimizes outliers.

ii. Do you agree that the geometric mean is good to use in this situation?

f. Try a few examples and form a conjecture as to whether the geometric mean is always smaller than the arithmetic mean, always larger than the arithmetic mean, or sometimes smaller and sometimes larger. Prove your conjecture in the case n = 2. (The general case is harder; we might do it later.)

g. Suppose you are considering buying a certain used car. You believe the car is worth \$8,000, and the seller believes it is worth \$12,000. A neutral party suggests you split the difference, by agreeing on the price \$10,000 (the arithmetic mean). You say that this isn't fair, because you would be paying 25% more than you think the car is worth, but the seller is getting only 16.7% less than he thinks it's worth. The seller says, "If you can come up with a fairer method, I'll accept it." What would you propose? (Consider the harmonic and geometric means, and give reasons why you think your choice is fair.)

h. We usually use the arithmetic mean (or a weighted version of it) for averaging grades to decide an overall grade. What would be arguments pro and con for using the median, harmonic mean, or geometric mean instead of the arithmetic mean?

i. Recall that the harmonic mean was used in one congressional reapportionment plan. The geometric mean was used in another; see (See "11. Hill's Method" at http://mathdl.maa.org/mathDL/46/?pa=content&sa=viewDocument&nodeId=3163)

j. In their book *Guesstimation*, Lawrence Weinstein and John. A. Adam suggest estimating a quantity by first estimating upper and lower bounds for the quantity, then taking the geometric mean of these upper and lower bounds. Do you think this is good advice? [Hint: Their goal is just to estimate the order of magnitude, not to get a precise answer.]

k. Weinstein and Adam also recommend a quick-and-dirty method of getting the *approximate* geometric mean of two numbers: If you have the numbers in scientific notation, average their coefficients and average their exponents. Does this seem reasonable? Why or why not? (It might help to try some examples and/or to think about the definition of geometric mean.) Note: This method only applies if the sum of the exponents is even. It the sum of exponents is odd, they recommend decreasing the exponent sum by one before dividing by 2, then multiplying the final number by 3. Why might this be reasonable?