

VIRTUAL RIEMANN-ROCH THEOREMS FOR ALMOST PERFECT OBSTRUCTION THEORIES

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ABSTRACT. This is the third in a series of works devoted to constructing virtual structure sheaves and K -theoretic invariants in moduli theory. The central objects of study are almost perfect obstruction theories, introduced by Y.-H. Kiem and the author as the appropriate notion in order to define invariants in K -theory for many moduli stacks of interest, including generalized K -theoretic Donaldson-Thomas invariants.

In this paper, we prove virtual Riemann-Roch theorems in the setting of almost perfect obstruction theory in both the non-equivariant and equivariant cases, including cosection localized versions. These generalize and remove technical assumptions from the virtual Riemann-Roch theorems of Fantechi-Göttsche and Ravi-Sreedhar. The main technical ingredients are a treatment of the equivariant K -theory and equivariant Gysin map of sheaf stacks and a formula for the virtual Todd class.

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1. INTRODUCTION

The main approach in obtaining enumerative invariants in algebraic geometry consists of setting up a moduli stack parametrizing the objects of interest and then constructing a fundamental class on the stack, against which one can integrate cohomology classes that capture geometric constraints in order to obtain numbers. However, it is typical that, while these moduli stacks often have an explicit expected dimension, they are very singular, far from equidimensional and do not behave well under deforming the moduli problem.

To rectify these phenomena, Li-Tian [LT98] and Behrend-Fantechi [BF97] developed the theory of virtual fundamental cycles, which are a generalization of the usual fundamental class for a smooth scheme. They have been instrumental in defining and investigating several algebro-geometric enumerative invariants of great importance, such as Gromov-Witten [Beh97],

Donaldson-Thomas [Tho00] and Pandharipande-Thomas [PT09] invariants, and are still one of the major components in modern enumerative geometry.

Their construction goes as follows: Any Deligne-Mumford stack X is equipped with its intrinsic normal cone \mathcal{C}_X which locally for an étale map $U \rightarrow X$ and a closed embedding $U \hookrightarrow M$ into a smooth scheme M is the quotient stack $[\mathcal{C}_{U/M}/T_M|_U]$ (cf. [BF97]). A perfect obstruction theory, defined in [BF97], is a morphism $\phi: E \rightarrow \mathbb{L}_X$ inducing an embedding of \mathcal{C}_X into the vector bundle stack $\mathcal{E}_X = h^1/h^0(E^\vee)$. Given this gadget, the virtual fundamental cycle is constructed as the intersection of the zero section $0_{\mathcal{E}_X}$ with \mathcal{C}_X

$$[X]^{\text{vir}} := 0_{\mathcal{E}_X}^![\mathcal{C}_X] \in A_*(X).$$

The existence of the virtual fundamental cycle is sufficient for constructing intersection theoretic and numerical enumerative invariants. Recently, there has been increased attention in such invariants that go beyond numbers or the intersection theory of cycles. Motivated by theoretical physics and geometric representation theory, invariants in K -theory are particularly desirable to have (see [Oko17, Oko19]).

When the moduli stack X admits a perfect obstruction theory with a global presentation $E = [E^{-1} \rightarrow E^0]$, where E^{-1}, E^0 are locally free sheaves on X , one can define the virtual structure sheaf, following [Lee04], as

$$[\mathcal{O}_X^{\text{vir}}] := 0_{E_1}^![\mathcal{O}_{C_1}] \in K_0(X)$$

where $E_1 = (E^{-1})^\vee$ and $C_1 = \mathcal{C}_X \times_{\mathcal{E}_X} E_1$.

However, there are many moduli stacks of interest which do not admit perfect obstruction theories or global presentations thereof, including moduli of simple complexes [Ina02] and modified blowups of stacks of semistable sheaves and perfect complexes on Calabi-Yau threefolds [KLS17, Sav20].

In earlier work [KS20a, KS20b], we developed a relaxed version of perfect obstruction theories, called almost perfect obstruction theories, which arise in the above moduli stacks, together with an extension of many of the usual tools, including virtual torus localization and cosection localization. An almost perfect obstruction theory admits an obstruction sheaf $\mathcal{O}b_X$, which is the analogue of the sheaf $h^1(E^\vee)$, and induces an embedding of the coarse moduli sheaf \mathfrak{c}_X of \mathcal{C}_X into $\mathcal{O}b_X$ enabling us to define the virtual structure sheaf of X as

$$(1.1) \quad [\mathcal{O}_X^{\text{vir}}] := 0_{\mathcal{O}b_X}^![\mathcal{O}_{\mathfrak{c}_X}] \in K_0(X).$$

To make sense of this expression, one of the main technical advances in [KS20a, KS20b] was the introduction of a K -theory of coherent sheaves on sheaf stacks and a Gysin map for sheaf stacks.

At the same time, the embedding $\mathfrak{c}_X \rightarrow \mathcal{O}b_X$ induces a cycle class $[\mathfrak{c}_X] \in A_*(\mathcal{O}b_X)$, defined by Chang-Li in [CL11], and hence, using the intersection theoretic Gysin map of Chang-Li for the sheaf stack $\mathcal{O}b_X$, a virtual fundamental cycle

$$(1.2) \quad [X]^{\text{vir}} := 0_{\mathcal{O}b_X}^![\mathfrak{c}_X] \in A_*(X).$$

It is a natural question whether the virtual structure sheaf (1.1) and the virtual fundamental cycle (1.2) induced by an almost perfect obstruction

theory are related. Since these generalize the structure sheaf and fundamental cycle of a smooth scheme, it is natural to expect that such a relation uses the Riemann-Roch transformation, whenever this is defined.

In the presence of a perfect obstruction theory, virtual Riemann-Roch theorems providing the desired relation have been established for algebraic spaces by Fantechi-Göttsche [FG10] and for quotient stacks by Ravi-Sreedhar [RS20]. Algebraic spaces and, more generally, quotient stacks are two important classes of algebraic stacks with a well-defined Riemann-Roch transformation.

In this paper, we prove a more general virtual Riemann-Roch theorem in the setting of almost perfect obstruction theory. Our main result can be summarized as follows.

Theorem. *Let G be an algebraic group, X an algebraic space with a G -action, Y a smooth scheme of pure dimension with a G -action and $f: X \rightarrow Y$ a G -equivariant morphism equipped with a G -equivariant almost perfect obstruction theory ϕ (cf. Definition 4.4).*

The virtual structure sheaf $[\mathcal{O}_X^{\text{vir}}]$ and virtual fundamental cycle $[X]^{\text{vir}}$ are then naturally G -equivariant (cf. Definition 4.5) and satisfy the equivariant Riemann-Roch formula

$$\tau_X^G([\mathcal{O}_X^{\text{vir}}]) = \text{td}^G(T_{X/Y}^{\text{vir}}) \cap [X]^{\text{vir}} \in A_*^G(X).$$

Here τ_X^G is the equivariant Riemann-Roch transformation of Edidin-Graham [EG00] and $\text{td}^G(T_{X/Y}^{\text{vir}})$ is defined by the formula (cf. Theorem 5.14)

$$\text{td}^G(T_{X/Y}^{\text{vir}}) = \frac{\text{td}^G(E_0)}{\text{td}^G(E_1)} \cdot \frac{\text{td}^G(f^*T_Y)}{\text{td}^G(\mathfrak{g})},$$

whenever X admits a G -invariant closed embedding into a smooth algebraic space with a G -action and $[E_0 \rightarrow E_1]$ is a G -equivariant two-term complex of locally free sheaves on X which gives a global presentation of the virtual tangent bundle of f (cf. Definition 5.12).

As a corollary, we obtain the following virtual Grothendieck-Riemann-Roch formula.

Corollary. *When f is proper, we have for any $V \in K_G^0(X)$*

$$\text{ch}(f_*(V \otimes [\mathcal{O}_X^{\text{vir}}])) \cdot \text{td}(T_Y) \cap [Y] = f_* \left(\text{ch}(V) \cdot \text{td}^G(T_{X/Y}^{\text{vir}}) \cap [X]^{\text{vir}} \right).$$

When the almost perfect obstruction theory ϕ admits a G -equivariant cosection $\mathcal{O}_b \rightarrow \mathcal{O}_X$, we also obtain cosection localized versions of the above statements.

Our virtual Riemann-Roch theorem is a generalization of [FG10, RS20] in multiple ways, as it only uses the existence of an almost perfect obstruction theory, and even in the case of a perfect obstruction theory requires a weaker notion of a global resolution of the two-term complex E in its definition.

Besides these, an essential technical input to the theorem of independent interest is the development of an equivariant version of the K -theory of coherent sheaves on sheaf stacks as well as an equivariant Gysin map. This

is necessary in order to define a natural equivariant structure on the virtual structure sheaf $[\mathcal{O}_X^{\text{vir}}]$.

When G is the trivial group, the theorem specializes to a non-equivariant virtual Riemann-Roch theorem for an almost perfect obstruction theory on the morphism $f: X \rightarrow Y$.

A noteworthy feature of the theorem is the explicit formula it provides for $\text{td}^G(T_{X/Y}^{\text{vir}})$ even though the virtual tangent bundle $T_{X/Y}^{\text{vir}}$ of f does not a priori make sense for an almost perfect obstruction theory. It would be interesting to investigate whether such a formula exists under more general conditions. This essentially boils down to representing the class $[T_{X/Y}] - [\mathcal{O}_X] \in K_0(X)$ by a class in $K^0(X)$. We formulate the most optimistic such scenario in Conjecture 5.4.

Finally, we mention that variants of the virtual Riemann-Roch theorem for Deligne-Mumford stacks with perfect obstruction theory have been studied in [Toe99, Jos07, Kha19], often utilizing stronger input from derived algebraic geometry. We expect our results to be compatible with these and apply to any Deligne-Mumford stack with a Riemann-Roch transformation $\tau_X: K_0(X) \rightarrow A_*(X)$ satisfying the usual properties.

Layout of the paper. §2 collects necessary background on the resolution property for stacks and Riemann-Roch theorems. In §3 we study the equivariant K -theory of coherent sheaves on sheaf stacks and their equivariant Gysin maps. §4 defines equivariant almost perfect obstruction theories and their induced virtual structure sheaves and fundamental cycles. In §5 we combine these ingredients to give a formula for the virtual Todd class and establish virtual Riemann-Roch theorems for almost perfect obstruction theories in the non-equivariant and equivariant cases. Finally, §6 generalizes these results in the presence of a cosection of the obstruction sheaf.

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Notation and conventions. Everything in this paper is over the field \mathbb{C} of complex numbers. All stacks are of finite type and Deligne-Mumford stacks are assumed to be separated.

G typically denotes a linear algebraic group.

Unless otherwise stated, $K_0(X)$ denotes the Grothendieck group of coherent sheaves on a stack X with \mathbb{Q} -coefficients and $K^0(X)$ denotes the Grothendieck group of locally free sheaves on X , also tensored with \mathbb{Q} . Similarly, $A_*(X)$ denotes the direct product of all Chow groups of a stack X with \mathbb{Q} -coefficients. When X admits an action by a group G , $K_0^G(X)$, $K^0_G(X)$ and $A_*^G(X)$ denote the corresponding G -equivariant groups.

If E is a locally free sheaf on a Deligne-Mumford stack X , we will use the term “vector bundle” to refer to its total space. If \mathcal{F} is a coherent sheaf on a Deligne-Mumford stack X , we will use the same letter to refer to the associated sheaf stack.

For a morphism $f : T \rightarrow X$ of stacks and a coherent sheaf \mathcal{F} on X , its pullback $f^*\mathcal{F}$ is sometimes denoted by $\mathcal{F}|_T$ when the map f is clear from context. The bounded derived category of coherent sheaves on a stack X is denoted by $D(X)$ and $\mathbb{L}_{X/Y} = L_{\bar{X}/Y}^{\geq -1}$ denotes the truncated cotangent complex for a morphism $X \rightarrow Y$.

2. BACKGROUND ON EQUIVARIANT GEOMETRY AND RIEMANN-ROCH

Throughout this section, G denotes a linear algebraic group. X will be a Deligne-Mumford stack with an action of G , Y is a smooth Artin stack of pure dimension with a G -action, and $f : X \rightarrow Y$ a G -equivariant morphism. For a detailed account of the foundations of group actions on stacks, we refer the reader to [Rom05].

2.1. The resolution property. The following definition explains several variants of the resolution property for stacks and introduces some useful terminology.

Definition 2.1. *We say that an Artin stack \mathcal{M} satisfies the resolution property if every coherent sheaf \mathcal{F} on \mathcal{M} is the quotient of a locally free sheaf.*

We say that \mathcal{M} satisfies the resolution property étale locally if there exists an étale cover $\rho : \mathcal{U} \rightarrow \mathcal{M}$, called an atlas for \mathcal{M} , such that \mathcal{U} satisfies the resolution property.

Finally, for a given coherent sheaf \mathcal{F} , we say that \mathcal{M} satisfies the resolution property étale locally for \mathcal{F} if there exists an étale cover $\rho_{\mathcal{F}} : \mathcal{U} \rightarrow \mathcal{M}$ such that the pullback $\rho_{\mathcal{F}}^\mathcal{F}$ is the quotient of a locally free sheaf on \mathcal{U} . In this case, the cover $\mathcal{U} \rightarrow \mathcal{M}$ is called an \mathcal{F} -atlas for \mathcal{M} .*

When $\mathcal{M} = [X/G]$ is a quotient stack, we can slightly specialize the notion of an atlas as follows.

Definition 2.2. *We say that X admits a G -equivariant atlas if there exists a G -equivariant étale cover $U \rightarrow X$ such that the stack $[U/G]$ satisfies the resolution property. We then say that $[U/G] \rightarrow [X/G]$ is an effective atlas for the quotient stack $\mathcal{M} = [X/G]$.*

Given a G -equivariant coherent sheaf \mathcal{F} on X , we say that X admits a G -equivariant atlas for \mathcal{F} if there exists a G -equivariant étale cover $\rho_{\mathcal{F}} : U \rightarrow X$ such that $\rho_{\mathcal{F}}^\mathcal{F}$ is the quotient of a G -equivariant locally free sheaf on U . We say in that case that $[U/G] \rightarrow [X/G]$ is an effective \mathcal{F} -atlas for \mathcal{M} .*

The following theorem is a collection of results in the literature which give conditions for the resolution property to hold on the nose or étale locally.

Theorem 2.3.

- (1) [Tho87, Tot04] *If X is a scheme that admits an ample family of G -equivariant line bundles, then $[X/G]$ satisfies the resolution property. In particular, this is true when X is quasi-affine.*
- (2) [Tot04, Gro17] *A quasi-compact quasi-separated Artin stack \mathcal{M} is of the form $[X/\mathrm{GL}_N]$, where X is a quasi-affine scheme, if and only if \mathcal{M} has affine stabilizers and satisfies the resolution property.*
- (3) [AHR20, Theorem 4.4] *If X is an algebraic space with a G -action and every point of X with closed G -orbit has reductive stabilizer, then X*

admits an affine (as a morphism) G -equivariant atlas $U \rightarrow X$ and hence $[U/G] \rightarrow [X/G]$ gives an affine effective atlas for $[X/G]$. In particular, this is true when $[X/G]$ is Deligne-Mumford.

- (4) [AHR20, Theorem 4.21] *If \mathcal{M} is an Artin stack whose closed points have reductive stabilizers, then \mathcal{M} satisfies the resolution property étale locally. In particular, any Deligne-Mumford stack satisfies the resolution property étale locally.*

2.2. Riemann-Roch for algebraic spaces. For every scheme X , Fulton [Ful98, Theorem 8.3] proves that there exists a Riemann-Roch transformation

$$(2.1) \quad \tau_X: K_0(X) \longrightarrow A_*(X),$$

which is a group homomorphism satisfying the following properties:

- (1) module homomorphism: for any $V \in K^0(X)$ and any $F \in K_0(X)$ one has $\tau_X(V \otimes F) = \text{ch}(V) \cap \tau_X(F)$;
- (2) Todd: if X is smooth, $\tau_X(\mathcal{O}_X) = \text{td}(T_X) \cap [X]$; hence for every $V \in K^0(X)$ one has $\tau_X(V \otimes \mathcal{O}_X) = \text{ch}(V) \cdot \text{td}(T_X) \cap [X]$;
- (3) covariance: for every proper morphism $f: X \rightarrow Y$ one has

$$f_* \circ \tau_X = \tau_Y \circ f_*: K_0(X) \rightarrow A_*(Y);$$

- (4) local complete intersection: if $f: X \rightarrow Y$ is an lci morphism, and $\alpha \in K_0(Y)$, then $f^*(\tau_Y(\alpha)) = (\text{td } T_f)^{-1} \cap \tau_X(f^*\alpha)$.

Motivated by (2), we define the Todd class of X by $\text{td}(X) := \tau_X(\mathcal{O}_X)$.

By [Gil84], Fulton's arguments extend to establish the existence of τ_X for any algebraic space X .

2.3. Riemann-Roch for quotient stacks. Suppose now that X is an algebraic space with an action of a linear algebraic group G with $\dim G = g$. Edidin-Graham [EG00] then construct an equivariant analogue of the Riemann-Roch transformation (2.1)

$$(2.2) \quad \tau_X^G: K_0^G(X) \longrightarrow A_*^G(X)$$

from the K -theory group of G -equivariant coherent sheaves on X to the G -equivariant Chow groups of X , defined in [EG98]. This can be equivalently viewed as a Riemann-Roch transformation for the quotient stack $[X/G]$, since there are natural isomorphisms

$$K_0^G(X) \cong K_0([X/G]), \quad A_*^G(X) \cong A_*([X/G]),$$

where $A_*([X/G])$ denotes the Chow groups defined by Kresch in [Kre99].

We review the construction of τ_X^G since it will be necessary in order to prove the main result of this paper later on.

Firstly, we recall the definition of $A_i^G(X)$.

Let V be a complex l -dimensional representation of G and let U be a G -invariant open subset of V such that G acts freely on U . The diagonal action on $X \times U$ is also free, so there is a quotient in the category of algebraic spaces $X \times U \rightarrow X \times^G U$, which in general may not be a scheme. The pair (V, U) is called an l -dimensional good pair for an integer j if $\text{codim}(U, V) > j$.

The i -th equivariant Chow group of X is defined as

$$A_i^G(X) := A_{i+l-g}(X \times^G U),$$

where (V, U) is an l -dimensional good pair for the integer $n - i$ and A_* denotes the usual Chow group. As usual, we write $A_*^G(X) = \prod_i A_i^G(X)$.

To define τ_X^G , we specify its i -th component $(\tau_X^G)_i$ mapping to $A_i^G(X)$. As above, let (V, U) be an l -dimensional good pair for the integer $n - i$. Write $j: U \rightarrow V$ for the open inclusion and denote by the same letter the open inclusion $X \times U \rightarrow X \times V$. Write $\pi_V: X \times V \rightarrow X$ for the projection. We also have a natural projection map $X \times^G (U \times V) \rightarrow X \times^G U$, which defines a vector bundle over $X \times^G U$. Then $(\tau_X^G)_i$ is defined by the commutative diagram

$$(2.3) \quad \begin{array}{ccccc} K_0^G(X \times V) & \xrightarrow{j^*} & K_0^G(X \times U) & \longrightarrow & K_0(X \times^G U) \\ \pi_X^* \uparrow & \nearrow \pi_X^* & \nearrow s_U & & \downarrow \left(\frac{\tau_{X \times^G U}}{\text{td}(X \times^G (U \times V))} \right)_{i+l-g} \\ K_0^G(X) & \xrightarrow{(\tau_X^G)_i} & A_{i+l-g}(X \times^G U), & & \end{array}$$

where the unlabelled top right morphism is the natural isomorphism, since the diagonal G -action on $X \times U$ is free.

2.4. Perfect obstruction theory, virtual structure sheaf and virtual fundamental cycle. We start with defining a perfect obstruction theory.

Definition 2.4. [BF97] *A perfect obstruction theory on a morphism $X \rightarrow Y$, where X is a Deligne-Mumford stack and Y a smooth, pure dimensional Artin stack, is a morphism*

$$\phi: E \longrightarrow \mathbb{L}_{X/Y}$$

in $D(X)$, where E is a perfect complex of amplitude $[-1, 0]$, satisfying that $h^{-1}(\phi)$ is surjective and $h^0(\phi)$ is an isomorphism.

We refer to the coherent sheaf $\mathcal{O}b_X := h^1(E^\vee)$ as the obstruction sheaf associated to the perfect obstruction theory ϕ .

A perfect obstruction theory ϕ induces a Cartesian diagram

$$(2.4) \quad \begin{array}{ccccc} \mathcal{C}_{X/Y} & \longrightarrow & \mathcal{N}_{X/Y} := h^1/h^0(\mathbb{L}_{X/Y}^\vee) & \xrightarrow{h^1/h^0(\phi^\vee)} & \mathcal{E} := h^1/h^0(E^\vee) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{c}_{X/Y} & \longrightarrow & \mathbf{n}_{X/Y} = h^1(\mathbb{L}_{X/Y}^\vee) & \xrightarrow{h^1(\phi^\vee)} & \mathcal{O}b_X = h^1(E^\vee) \end{array}$$

where $\mathcal{C}_{X/Y}$ and $\mathcal{N}_{X/Y}$ are the intrinsic normal cone and intrinsic normal sheaf of X over Y respectively, while $\mathbf{c}_{X/Y}$ and $\mathbf{n}_{X/Y}$ are their coarse moduli sheaves. All the horizontal arrows are closed embeddings.

Assuming that E admits a global resolution by vector bundles on X so that we may write $E = [E^{-1} \rightarrow E^0]$, we have that $\mathcal{E} = [E_1/E_0]$, where we denote $E_0 = (E^0)^\vee, E_1 = (E^{-1})^\vee$. In this case, the above diagram can be

augmented by the Cartesian square

$$(2.5) \quad \begin{array}{ccc} C_1 & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ \mathcal{C}_{X/Y} & \longrightarrow & \mathcal{E} = [E_1/E_0]. \end{array}$$

We may then define the virtual fundamental cycle $[X]^{\text{vir}} \in A_*(X)$ and virtual structure sheaf $[\mathcal{O}_X^{\text{vir}}] \in K_0(X)$ using the intersection-theoretic and K -theoretic Gysin map as follows.

Definition 2.5. [BF97, LT98] *The virtual fundamental cycle of X is defined by $[X]^{\text{vir}} = 0_{E_1}^! [C_1] \in A_*(X)$.*

Definition 2.6. [Lee04] *The virtual structure sheaf of X is defined by $[\mathcal{O}_X^{\text{vir}}] = 0_{E_1}^! [C_1] \in K_0(X)$.*

2.5. Virtual Riemann-Roch for schemes with perfect obstruction theory. In [FG10], Fantechi and Göttsche prove a virtual Riemann-Roch theorem for quasiprojective schemes X endowed with a perfect obstruction theory $\phi: E \rightarrow \mathbb{L}_X$, under the assumption that E admits a global resolution $[E^{-1} \rightarrow E^0]$. To state their theorem, we define the K -theoretic virtual tangent bundle of X by the formula

$$T_X^{\text{vir}} = [E_0] - [E_1] \in K^0(X),$$

where as above $E_0 = (E^0)^\vee, E_1 = (E^{-1})^\vee$.

Theorem 2.7. [FG10, Theorem 3.3, Lemma 3.5] *We have*

$$(2.6) \quad \tau_X([\mathcal{O}_X^{\text{vir}}]) = \text{td}(T_X^{\text{vir}}) \cap [X]^{\text{vir}}.$$

As a corollary, if $f: X \rightarrow Y$ is a proper morphism from X to a smooth, pure dimensional scheme Y and $V \in K^0(X)$, we obtain the virtual Grothendieck-Riemann-Roch formula

$$\text{ch}(f_*(V \otimes [\mathcal{O}_X^{\text{vir}}])) \cdot \text{td}(T_Y) \cap [Y] = f_*(\text{ch}(V) \cdot \text{td}(T_X^{\text{vir}}) \cap [X]^{\text{vir}}).$$

2.6. Equivariant virtual Riemann-Roch. In [RS20], Ravi and Sreedhar prove an equivariant virtual Riemann-Roch theorem. A statement of their result is as follows.

Theorem 2.8. [RS20, Theorem 1.1] *Let X, Y be G -schemes and $f: X \rightarrow Y$ be a G -equivariant morphism such that Y is smooth, G -equivariantly connected and pure dimensional. Suppose that there exists a G -invariant closed embedding $X \rightarrow M$ for some smooth G -scheme M such that for any l -dimensional good pair for the integer $n - i$ the quotient $X \times^G U$ is a scheme. Then for any equivariant perfect relative obstruction theory $E \rightarrow \mathbb{L}_{X/Y}$ on X with respect to Y which admits a global resolution $E = [E^{-1} \rightarrow E^0]$, we have*

$$(2.7) \quad \tau_X([\mathcal{O}_X^{\text{vir}}]) = \text{td}^G(T_{X/Y}^{\text{vir}}) \cap [X]^{\text{vir}} \in A_*^G(X),$$

where $\text{td}^G(T_{X/Y}^{\text{vir}}) = [E_0] - [E_1] + [f^*T_Y] - [\mathfrak{g}] \in K_G^0(X)$ and $E_0 = (E^0)^\vee, E_1 = (E^{-1})^\vee$ are G -equivariant locally free sheaves on X .

3. EQUIVARIANT K -THEORY AND GYSIN MAPS ON SHEAF STACKS

Throughout this section, G denotes a linear algebraic group, X is a Deligne-Mumford stack with an action of G , Y is a smooth Artin stack of pure dimension with a G -action, and $f: X \rightarrow Y$ a G -equivariant morphism. \mathcal{F} will be a G -equivariant coherent sheaf on X .

In [KS20a], the authors introduced the notion of coherent sheaves on the sheaf stack \mathcal{F} , defined the K -theory of \mathcal{F} and constructed a Gysin map $0_{\mathcal{F}}^!$ of \mathcal{F} , which made it possible to construct the virtual structure sheaf $[\mathcal{O}_X^{\text{vir}}] \in K_0(X)$ for a Deligne-Mumford stack X equipped with an almost perfect obstruction theory. This was performed in the non-equivariant case where G is the trivial group.

In this section, we recall the main constructions that will be necessary from [KS20a] and at the same time develop their generalization to the equivariant context.

3.1. Sheaf stacks, local charts and common roofs. We may associate a sheaf stack to a coherent sheaf on X .

Definition 3.1. (Sheaf stack) *The sheaf stack associated to \mathcal{F} is the stack that to every morphism $\rho: T \rightarrow X$ from a scheme T associates the set of sections $\Gamma(T, \rho^*\mathcal{F})$.*

By abuse of notation, we denote by \mathcal{F} the sheaf stack associated to a coherent sheaf \mathcal{F} on X .

When \mathcal{F} is a G -equivariant sheaf, the sheaf stack \mathcal{F} naturally admits an induced G -action: Let $\sigma: X \times G \rightarrow X$ be the G -action morphism on X and $\pi_X: X \times G \rightarrow X$ the first projection map. By definition, \mathcal{F} is a G -equivariant sheaf if there exists an isomorphism

$$\mu: \sigma^*\mathcal{F} \longrightarrow \pi_X^*\mathcal{F}.$$

To specify the G -action on the associated sheaf stack, we need a morphism

$$\nu: \mathcal{F} \times G \longrightarrow \mathcal{F}$$

satisfying the usual axioms of a group action.

Consider a T -point of $\mathcal{F} \times G$. This consists of the data

$$\rho: T \rightarrow X, \quad s \in \Gamma(T, \rho^*\mathcal{F}), \quad g: T \rightarrow G.$$

The image of this point under ν is then the T -point of \mathcal{F} consisting of the data

$$\nu(\rho): T \xrightarrow{\rho \times g} X \times G \xrightarrow{\sigma} X, \quad \nu(s) \in \Gamma(T, \nu(\rho)^*\mathcal{F}),$$

where the section $\nu(s)$ is defined as the composition

$$\nu(s): \mathcal{O}_T \xrightarrow{s} \rho^*\mathcal{F} = (\rho \times g)^*\pi_1^*\mathcal{F} \xrightarrow{(\rho \times g)^*\mu^{-1}} (\rho \times g)^*\sigma^*\mathcal{F} = \nu(\rho)^*\mathcal{F}.$$

It is routine to check that ν defines a G -action and moreover the projection morphism $\mathcal{F} \rightarrow X$ is G -equivariant.

A sheaf stack is not algebraic in general and we need an appropriate notion of local charts for geometric constructions.

Definition 3.2. (Local chart) *A local chart $Q = (U, \rho, E, r_E)$ for the sheaf stack \mathcal{F} consists of*

- (1) an étale morphism $\rho : U \rightarrow X$ from a scheme U , and
- (2) a surjective homomorphism $r_E : E \rightarrow \rho^* \mathcal{F} = \mathcal{F}|_U$ of coherent sheaves on U from a locally free sheaf E on U .

We will call U the base of the chart Q . If U is affine and E is free, then the local chart $Q = (U, \rho, E, r_E)$ is called affine.

When \mathcal{F} is G -equivariant, we say that the chart Q is G -equivariant if U is a G -scheme, ρ is G -equivariant, E is a G -equivariant locally free sheaf and the homomorphism r_E is G -equivariant.

Definition 3.3. (Morphism between local charts) Let $Q = (U, \rho, E, r_E)$ and $Q' = (U', \rho', E', r_{E'})$ be two local charts for \mathcal{F} . A morphism $\gamma : Q \rightarrow Q'$ is the pair (ρ_γ, r_γ) of an étale morphism $\rho_\gamma : U \rightarrow U'$ and a surjection $r_\gamma : E \rightarrow \rho_\gamma^* E'$ of locally free sheaves, such that the diagrams

$$(3.1) \quad \begin{array}{ccc} U & \xrightarrow{\rho_\gamma} & U' \\ & \searrow \rho & \downarrow \rho' \\ & & X \end{array} \quad \begin{array}{ccc} E & \xrightarrow{r_\gamma} & \rho_\gamma^* E' \\ & \searrow r_E & \downarrow \rho_\gamma^* r_{E'} \\ & & \mathcal{F}|_U \end{array}$$

are commutative.

We say that Q is a restriction of Q' and write $Q = Q'|_U$ if $E = \rho_\gamma^* E'$ and r_γ is the identity morphism.

For a G -equivariant sheaf \mathcal{F} and equivariant local charts Q, Q' , the morphism $\gamma : Q \rightarrow Q'$ is called G -equivariant when ρ_γ and r_γ are equivariant.

The notion of a common roof enables us compare two local charts on \mathcal{F} with the same base $\rho : U \rightarrow X$.

Definition 3.4. (Common roof) Let $r : \mathcal{G} \rightarrow \mathcal{F}$ and $r' : \mathcal{G}' \rightarrow \mathcal{F}$ be two surjective homomorphisms of coherent sheaves on a scheme U . Their fiber product is defined by

$$(3.2) \quad \mathcal{G} \times_{\mathcal{F}} \mathcal{G}' := \ker \left(\mathcal{G} \oplus \mathcal{G}' \xrightarrow{(r, -r')} \mathcal{F} \oplus \mathcal{F} \xrightarrow{\pm} \mathcal{F} \right)$$

and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{G} \times_{\mathcal{F}} \mathcal{G}' & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{G}' & \longrightarrow & \mathcal{F} \end{array}$$

of surjective homomorphisms, which is universal among such diagrams of surjective homomorphisms in the obvious sense.

Given two charts $Q = (U, \rho, E, r_E)$ and $Q' = (U, \rho, E', r_{E'})$ with the same quasi-projective base U , we can pick a surjective homomorphism

$$(3.3) \quad W \longrightarrow E \times_{\mathcal{F}|_U} E'$$

from a locally free sheaf W . Denoting the induced surjection $W \rightarrow \mathcal{F}|_U$ by r_W , we obtain a local chart (U, ρ, W, r_W) with natural morphisms to Q and Q' , which we call a common roof of Q and Q' .

When \mathcal{F} and Q, Q' are G -equivariant, we obtain an equivariant common roof by requiring that W and the morphism (3.3) are G -equivariant.

More generally, given two charts $Q = (U, \rho, E, r_E)$ and $Q' = (U', \rho', E', r_{E'})$ of the sheaf stack \mathcal{F} , we let $V = U \times_X U'$ and have two local charts $Q|_V$ and $Q'|_V$ with the same base. A common roof of $Q|_V$ and $Q'|_V$ is called a *common roof* of Q and Q' .

3.2. Coherent sheaves on a sheaf stack \mathcal{F} . A *coherent sheaf* \mathcal{A} on \mathcal{F} is an assignment to every local chart $Q = (U, \rho, E, r_E)$ of a coherent sheaf \mathcal{A}_Q on the scheme E (in the étale topology) such that for every morphism $\gamma: Q \rightarrow Q'$ between local charts there exists an isomorphism

$$(3.4) \quad r_\gamma^* (\rho_\gamma^* \mathcal{A}_{Q'}) \longrightarrow \mathcal{A}_Q$$

which satisfies the usual compatibilities for composition of morphisms. Note that we abusively write $\rho_\gamma^* \mathcal{A}_{Q'}$ for the pullback of $\mathcal{A}_{Q'}$ to $\rho_\gamma^* E'$ via the morphism of bundles $\rho_\gamma^* E' \rightarrow E'$ induced by ρ_γ . A *quasicoherent sheaf* on a sheaf stack is defined likewise.

A homomorphism $f: \mathcal{A} \rightarrow \mathcal{B}$ of (quasi)coherent sheaves on \mathcal{F} is the data of a homomorphism $f_Q: \mathcal{A}_Q \rightarrow \mathcal{B}_Q$ of (quasi)coherent sheaves on E for each local chart $Q = (U, \rho, E, r_E)$ such that for every morphism $\gamma: Q \rightarrow Q'$ of local charts, the diagram

$$\begin{array}{ccc} r_\gamma^* (\rho_\gamma^* \mathcal{A}_{Q'}) & \longrightarrow & \mathcal{A}_Q \\ f_{Q'} \downarrow & & \downarrow f_Q \\ r_\gamma^* (\rho_\gamma^* \mathcal{B}_{Q'}) & \longrightarrow & \mathcal{B}_Q \end{array}$$

is commutative where the horizontal arrows are as in (3.4). We say that a homomorphism $f: \mathcal{A} \rightarrow \mathcal{B}$ is an isomorphism if f_Q is an isomorphism for each local chart Q .

We would now like to define G -equivariant sheaves on \mathcal{F} when \mathcal{F} and X admit compatible G -actions. In order to do so, we need the following lemma, which says that our definition of a sheaf coincides with the usual definition of a sheaf on a stack.

Lemma 3.5. *A (quasi)coherent sheaf \mathcal{A} on a sheaf stack \mathcal{F} is equivalent to an assignment to every T -point $\tau: T \rightarrow \mathcal{F}$ of a (quasi)coherent sheaf \mathcal{A}_τ on T (in the étale topology) such that whenever we have a composition*

$$\tau: T \xrightarrow{f} T' \xrightarrow{\tau'} \mathcal{F}$$

there exists an isomorphism

$$(3.5) \quad f^* \mathcal{A}_{\tau'} \longrightarrow \mathcal{A}_\tau.$$

These isomorphisms satisfy the usual compatibility conditions.

Proof. This is a descent statement. Clearly if we have such an assignment \mathcal{A}_τ for every T -point $\tau: T \rightarrow \mathcal{F}$, then for a local chart $Q = (U, \rho, E, r_E)$ we get an E -point of \mathcal{F} given by the composition

$$\tau_Q: E \xrightarrow{r_E} \mathcal{F}|_U = \rho^* \mathcal{F} \longrightarrow \mathcal{F},$$

where the last morphism is the natural open inclusion.

We can thus set $\mathcal{A}_Q := \mathcal{A}_{\tau_Q}$. For a morphism $\gamma: Q \rightarrow Q'$ of local charts, we obtain an induced composition by (3.1)

$$(3.6) \quad \tau_Q: E \xrightarrow{r_\gamma} E' \xrightarrow{\tau_{Q'}} \mathcal{F}$$

where by slight abuse of notation r_γ is the map of vector bundles fitting in the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{r_\gamma} & E' \\ \downarrow & & \downarrow \\ U & \xrightarrow{\rho_\gamma} & U'. \end{array}$$

The isomorphism (3.5) corresponding to the composition (3.6) can be written equivalently as

$$r_\gamma^*(\rho_\gamma^* \mathcal{A}_{Q'}) \longrightarrow \mathcal{A}_Q.$$

Hence we obtain a (quasi)coherent sheaf on \mathcal{F} .

Conversely, suppose we are given a (quasi)coherent sheaf \mathcal{A} on \mathcal{F} and a T -point $\tau: T \rightarrow \mathcal{F}$ which consists of the data of a morphism $\rho: T \rightarrow X$ and a section

$$s: \mathcal{O}_T \longrightarrow \rho^* \mathcal{F}.$$

We need to specify the assignment of a sheaf \mathcal{A}_τ on T . By the pullback formalism developed in [KS20b, Section 3], there exists a functorial pullback sheaf $\rho^* \mathcal{A}$ on the sheaf stack $\rho^* \mathcal{F}$ over T . By replacing X and \mathcal{F} with T and $\rho^* \mathcal{F}$ we may thus assume that $T = X$ and $\rho = \text{id}_X$.

Since \mathcal{F} is a coherent sheaf on X , there is an affine étale cover $\{\rho_\alpha: X_\alpha \rightarrow X\}$ and local charts $Q_\alpha = (X_\alpha, \rho_\alpha, E_\alpha, r_\alpha)$ for \mathcal{F} . By possibly shrinking X_α , we may assume that E_α is free and thus the section $s_\alpha := s|_{X_\alpha}$ of $\mathcal{F}|_{X_\alpha}$ lifts to a section of E_α

$$(3.7) \quad \begin{array}{ccc} & & E_\alpha \\ & \nearrow \tilde{s}_\alpha & \downarrow r_\alpha \\ \mathcal{O}_{X_\alpha} & \xrightarrow{s|_{X_\alpha}} & \mathcal{F}|_{X_\alpha}. \end{array}$$

In particular, we obtain a morphism of schemes $\tilde{s}_\alpha: X_\alpha \rightarrow E_\alpha$ and can define $\mathcal{A}_\alpha := (\tilde{s}_\alpha)^* \mathcal{A}_{Q_\alpha}$. Observe that \mathcal{A}_α does not depend on the choice of lift: For any two choices $\tilde{s}_\alpha, \tilde{s}'_\alpha$, their difference $\tilde{s}'_\alpha - \tilde{s}_\alpha$ factors through a section k_α of $\ker(r_\alpha)$. We thus have commutative diagrams

$$(3.8) \quad \begin{array}{ccc} E_\alpha \oplus \ker(r_\alpha) & & E_\alpha \oplus \ker(r_\alpha) \\ \downarrow \pi_1 & & \downarrow + \\ \tilde{s}_\alpha \oplus k_\alpha \nearrow & E_\alpha & \tilde{s}'_\alpha \nearrow E_\alpha \\ \downarrow r_\alpha & \downarrow r_\alpha & \downarrow r_\alpha \\ \mathcal{O}_{X_\alpha} \xrightarrow{s|_{X_\alpha}} \mathcal{F}|_{X_\alpha}, & & \mathcal{O}_{X_\alpha} \xrightarrow{s|_{X_\alpha}} \mathcal{F}|_{X_\alpha}. \end{array}$$

Since $\tilde{r}_\alpha := r_\alpha \circ \pi_1 = r_\alpha \circ +$, by taking a surjection $F_\alpha \rightarrow \ker(r_\alpha)$ with F_α free, we get a local chart $\tilde{Q}_\alpha = (X_\alpha, \rho_\alpha, E_\alpha \oplus F_\alpha, \tilde{r}_\alpha)$ and two morphisms

$\gamma, \gamma': \tilde{Q}_\alpha \rightarrow Q_\alpha$ determined by the morphisms $E_\alpha \oplus F_\alpha \xrightarrow{\pi_1} E_\alpha$ and $E_\alpha \oplus F_\alpha \xrightarrow{+} E_\alpha$.

Lifting k_α to any section \tilde{k}_α of F_α , we get new commutative diagrams

$$(3.9) \quad \begin{array}{ccc} & E_\alpha \oplus F_\alpha & \\ & \downarrow \pi_1 & \\ \tilde{s}_\alpha \oplus \tilde{k}_\alpha \nearrow & E_\alpha & \\ \tilde{s}_\alpha \nearrow & \downarrow r_\alpha & \\ \mathcal{O}_{X_\alpha} & \xrightarrow{s|_{X_\alpha}} & \mathcal{F}|_{X_\alpha}, \end{array} \quad \begin{array}{ccc} & E_\alpha \oplus F_\alpha & \\ & \downarrow + & \\ \tilde{s}_\alpha \oplus \tilde{k}_\alpha \nearrow & E_\alpha & \\ \tilde{s}'_\alpha \nearrow & \downarrow r_\alpha & \\ \mathcal{O}_{X_\alpha} & \xrightarrow{s|_{X_\alpha}} & \mathcal{F}|_{X_\alpha}. \end{array}$$

The isomorphisms (3.4) now immediately imply that there is a natural isomorphism $\pi_1^* \mathcal{A}_{Q_\alpha} \simeq +^* \mathcal{A}_{Q_\alpha}$ of sheaves on the vector bundle $E_\alpha \oplus F_\alpha$. Pulling back via $\tilde{s}_\alpha \oplus \tilde{k}_\alpha$, we see that \mathcal{A}_α is defined up to canonical isomorphism.

Using the same reasoning and common roofs for overlaps, a standard descent argument implies that the sheaves \mathcal{A}_α glue canonically to define a sheaf on X , which by definition is the sheaf \mathcal{A}_τ . The fact that \mathcal{A}_τ satisfies the functoriality condition (3.5) is routine. We leave the details to the reader. \square

We are now in position to define G -equivariant sheaves on sheaf stacks.

Definition 3.6. *Let X be a Deligne-Mumford stack with a G -action and \mathcal{F} a G -equivariant coherent sheaf on X , with G -action determined by the morphism $\nu: \mathcal{F} \times G \rightarrow \mathcal{F}$.*

We say that a (quasi)coherent sheaf \mathcal{A} on \mathcal{F} is G -equivariant if there exists an isomorphism

$$\nu^* \mathcal{A} \longrightarrow \pi_{\mathcal{F}}^* \mathcal{A}$$

satisfying the axioms of a group action, where $\pi_{\mathcal{F}}: \mathcal{F} \times G \rightarrow \mathcal{F}$ denotes the projection. Morphisms between G -equivariant sheaves are defined in the obvious way.

By Lemma 3.5 the meaning of pullback in the definition is unambiguous. Observe that, as expected, if \mathcal{A} is a G -equivariant sheaf on \mathcal{F} and $Q = (U, \rho, E, r_E)$ is a G -equivariant local chart, then \mathcal{A}_Q is a G -equivariant sheaf on the G -scheme E .

Exact sequences and the K -group $K_0(\mathcal{F})$ were defined in [KS20a] as follows.

Definition 3.7. (Short exact sequence) *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be coherent sheaves on the sheaf stack \mathcal{F} . A sequence*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

of homomorphisms of coherent sheaves on \mathcal{F} is exact if for every local chart $Q = (U, \rho, E, r_E)$ on \mathcal{F} the sequence

$$0 \longrightarrow \mathcal{A}_Q \longrightarrow \mathcal{B}_Q \longrightarrow \mathcal{C}_Q \longrightarrow 0$$

is an exact sequence of coherent sheaves on the scheme E .

Note that the morphism

$$E \xrightarrow{r_\gamma} \rho_\gamma^* E' = U \times_{U'} E' \longrightarrow E'$$

is smooth and hence flat. We can likewise define the *kernel* and *cokernel* of a homomorphism $f : \mathcal{A} \rightarrow \mathcal{B}$ of coherent sheaves on the sheaf stack \mathcal{F} . Thus coherent and quasicoherent sheaves on \mathcal{F} form abelian categories

$$\mathrm{Coh}(\mathcal{F}) \subset \mathrm{QCoh}(\mathcal{F}).$$

In the G -equivariant case, we analogously obtain abelian categories

$$\mathrm{Coh}^G(\mathcal{F}) \subset \mathrm{QCoh}^G(\mathcal{F}),$$

with objects G -equivariant sheaves and G -equivariant morphisms between them.

Definition 3.8. *The K -group of coherent sheaves on \mathcal{F} is the group $K_0(\mathcal{F})$ generated by the isomorphism classes $[\mathcal{A}]$ of coherent sheaves \mathcal{A} on \mathcal{F} , with relations generated by $[\mathcal{B}] = [\mathcal{A}] + [\mathcal{C}]$ for every short exact sequence*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0.$$

When \mathcal{F} admits a G -action, we have the corresponding notion of the G -equivariant K -group $K_0^G(\mathcal{F})$.

In other words, $K_0(\mathcal{F})$ and $K_0^G(\mathcal{F})$ are the Grothendieck groups of the abelian categories $\mathrm{Coh}(\mathcal{F})$ and $\mathrm{Coh}^G(\mathcal{F})$ respectively.

If $\mathcal{F} = E$ is locally free, so that \mathcal{F} is an algebraic stack, then the above definitions recover the standard notions of short exact sequences and $K_0(E)$ and $K_0^G(E)$ for the vector bundle E .

3.3. Equivariant Gysin maps for sheaf stacks in K -theory. Let $Q = (U, \rho, E, r_E)$ be a local chart for the sheaf stack \mathcal{F} and denote the vector bundle projection map $E \rightarrow U$ by π_E . The tautological section of the pullback $\pi_E^* E$ induces an associated Koszul complex

$$\mathcal{K}(E) := \wedge^\bullet \pi_E^* E^\vee$$

that resolves the structure sheaf \mathcal{O}_U of the zero section of π_E .

Definition 3.9. *For any local chart $Q = (U, \rho, E, r_E)$ and a coherent sheaf \mathcal{A} on \mathcal{F} , the i -th Koszul homology sheaf*

$$\mathcal{H}_Q^i(\mathcal{A}) \in \mathrm{Coh}(U)$$

of \mathcal{A} with respect to Q is defined as the homology of the complex

$$\mathcal{K}(E) \otimes_{\mathcal{O}_E} \mathcal{A}_Q = \wedge^\bullet \pi_E^* E^\vee \otimes_{\mathcal{O}_E} \mathcal{A}_Q$$

in degree $-i$.

Using common roofs and the standard descent theory for coherent sheaves on algebraic stacks, the following is proven in [KS20a].

Theorem-Definition 3.10. *Let \mathcal{A} be a coherent sheaf on a sheaf stack \mathcal{F} over a Deligne-Mumford stack X . The coherent sheaves $\mathcal{H}_Q^i(\mathcal{A}) \in \mathrm{Coh}(U)$ glue canonically to a coherent sheaf $\mathcal{H}_X^i(\mathcal{A}) \in \mathrm{Coh}(X)$ on X , which is defined to be the i -th Koszul homology sheaf of \mathcal{A} .*

The Koszul homology sheaves of \mathcal{A} can then be used to define a K -theoretic Gysin map $0_{\mathcal{F}}^!: K_0(\mathcal{F}) \rightarrow K_0(X)$.

Definition 3.11. (K -theoretic Gysin map) *The K -theoretic Gysin map is defined by the formula*

$$(3.10) \quad 0_{\mathcal{F}}^!: K_0(\mathcal{F}) \rightarrow K_0(X), \quad 0_{\mathcal{F}}^![\mathcal{A}] = \sum_{i \geq 0} (-1)^i [\mathcal{H}_{\mathcal{K}}^i(\mathcal{A})] \in K_0(X)$$

where \mathcal{A} is a coherent sheaf on \mathcal{F} .

For the purposes of this paper, we need equivariant generalizations of these constructions. These are provided by the following theorem when X is an algebraic space.

Theorem 3.12. *Suppose that \mathcal{F} is a G -equivariant sheaf over an algebraic space X with a G -action. For any G -equivariant coherent sheaf \mathcal{A} on the sheaf stack \mathcal{F} , the Koszul homology sheaves $\mathcal{H}_{\mathcal{K}}^i(\mathcal{A})$ admit natural equivariant lifts in $\text{Coh}^G(X)$ and there exists a G -equivariant K -theoretic Gysin map*

$$(3.11) \quad 0_{\mathcal{F}}^{!,G}: K_0^G(\mathcal{F}) \rightarrow K_0^G(X)$$

defined by the formula (3.10).

Proof. Let \mathcal{A} be a G -equivariant sheaf on the sheaf stack \mathcal{F} . Let U be a quasi-affine scheme on which G acts freely and write $\pi_X: X \times U \rightarrow X$ for the projection map. The diagonal action of G on $X \times U$ is free and thus by Theorem 2.3(3), $X \times U$ admits an affine G -equivariant atlas $V \rightarrow X \times U$. In particular, we obtain a smooth affine G -equivariant surjection $\rho: V \rightarrow X$.

Consider the commutative diagrams with Cartesian rhombi

$$(3.12)$$

The diagram consists of two Cartesian rhombi. The left rhombus has vertices $\rho^*\mathcal{F}$ (top-left), $\rho^*\mathcal{F} \times_{\mathcal{F}} \rho^*\mathcal{F}$ (top-right), $V \times_X V$ (middle), and V (bottom). Arrows are π_1 (top-left to top-right), π_2 (top-right to middle), ρ (middle to bottom), and ρ (top-left to bottom). The right rhombus has vertices $\rho^*\mathcal{F}$ (top-left), $\rho^*\mathcal{F} \times_{\mathcal{F}} \rho^*\mathcal{F}$ (top-right), \mathcal{F} (middle), and V (bottom). Arrows are π_1 (top-left to top-right), π_2 (top-right to middle), ρ (middle to bottom), and ρ (top-left to bottom). Both V nodes are connected to X at the bottom by arrows labeled ρ .

By smooth descent, we may equivalently think of \mathcal{A} as a G -equivariant sheaf $\rho^*\mathcal{A}$ on the sheaf stack $\rho^*\mathcal{F}$ together with a G -equivariant descent datum $\pi_1^*\rho^*\mathcal{A} \simeq \pi_2^*\rho^*\mathcal{A}$, an isomorphism between the pullbacks of $\rho^*\mathcal{A}$ via the two projections π_1, π_2 to the sheaf stack $\rho^*\mathcal{F} \times_{\mathcal{F}} \rho^*\mathcal{F}$.

Since V satisfies the resolution property, there exists a G -equivariant locally free sheaf E on V and a surjection $r: E \rightarrow \rho^*\mathcal{F}$, so that we have a surjective local chart $Q_V = (V, \text{id}_V, E, r)$ for $\rho^*\mathcal{F}$ on V . We then obtain G -equivariant Koszul homology sheaves $\mathcal{H}_{\mathcal{K}}^i(\rho^*\mathcal{A})$ on V defined as the

homology sheaves of the G -equivariant complex

$$\mathcal{K}(E) \otimes_{\mathcal{O}_E} (\rho^* \mathcal{A})_{Q_V}.$$

Pulling back, we obtain surjective local charts $Q_i = (V \times_X V, \text{id}, \pi_i^* E, \pi_i^* r)$ for the sheaf stack $\pi_i^* \rho^* \mathcal{F} \simeq \rho^* \mathcal{F} \times_{\mathcal{F}} \rho^* \mathcal{F}$ where $i = 1, 2$. Since X is separated and V can be taken to be affine, $V \times_X V$ is also an affine G -scheme, which by Theorem 2.3(1) satisfies the resolution property. In particular, there exists a G -equivariant surjection

$$W \longrightarrow \pi_1^* E \times_{\rho^* \mathcal{F} \times_{\mathcal{F}} \rho^* \mathcal{F}} \pi_2^* E$$

where W is a G -equivariant locally free sheaf. By Definition 3.4, this gives a local chart $Q_W = (V \times_X V, \text{id}, W, r_W)$ which is a G -equivariant common roof for Q_1 and Q_2 .

Using this common roof and the canonical nature of the Koszul homology sheaves, the descent datum $\pi_1^* \rho^* \mathcal{A} \simeq \pi_2^* \rho^* \mathcal{A}$ induces a G -equivariant isomorphism of coherent sheaves on V

$$(3.13) \quad \pi_1^* \mathcal{H}_{\mathcal{K}}^i(\rho^* \mathcal{A}) \simeq \mathcal{H}_{\mathcal{K}}^i(\pi_1^* \rho^* \mathcal{A}) \simeq \mathcal{H}_{\mathcal{K}}^i(\pi_2^* \rho^* \mathcal{A}) \simeq \pi_2^* \mathcal{H}_{\mathcal{K}}^i(\rho^* \mathcal{A}).$$

It follows by smooth descent that the Koszul homology sheaves $\mathcal{H}_{\mathcal{K}}^i(\rho^* \mathcal{A})$ descend to G -equivariant sheaves $\mathcal{H}_{\mathcal{K}}^i(\mathcal{A})$ on X . These are independent of the choices of smooth affine equivariant atlas $\rho: V \rightarrow X$ and surjection $E \rightarrow \rho^* \mathcal{F}$, since for any other choices $\rho': V' \rightarrow X$ and $E' \rightarrow (\rho')^* \mathcal{F}$, we may work with the affine equivariant atlas

$$(3.14) \quad V \times_X V' \rightarrow X$$

and any equivariant surjection

$$(3.15) \quad W' \rightarrow \pi_1^* E \times_{\rho^* \mathcal{F} \times_{\mathcal{F}} (\rho')^* \mathcal{F}} \pi_2^* E'$$

from an equivariant locally free sheaf W' , where $\pi_1: V \times_X V' \rightarrow V$ and $\pi_2: V \times_X V' \rightarrow V'$ are the two projection maps. It is routine to check that the G -equivariant sheaves $\mathcal{H}_{\mathcal{K}}^i(\mathcal{A})$ are the same as the ones obtained using the atlas (3.14) and surjection from a locally free sheaf (3.15), so they are independent of any choices made.

Forgetting the equivariant structure, as coherent sheaves on X , these are isomorphic to the previously defined Koszul homology sheaves $\mathcal{H}_{\mathcal{K}}^i(\mathcal{A})$. This is because for any local chart for \mathcal{F} we obtain by pullback an induced local chart for $\rho^* \mathcal{F}$. By the definition and canonicity of the Koszul homology sheaves, it is then immediate that the Koszul homology sheaves $\mathcal{H}_{\mathcal{K}}^i(\rho^* \mathcal{A})$ are alternatively obtained using a cover by these induced local charts and the corresponding descent datum is compatible with (3.13) (in fact it is its pullback to the étale cover of V given by the induced local charts for $\rho^* \mathcal{F}$). This observation concludes the proof. \square

When X is Deligne-Mumford, we make the extra assumption of the existence of an affine G -equivariant atlas for \mathcal{F} .

Theorem 3.13. *Suppose that X is a Deligne-Mumford stack with a G -action and an affine G -equivariant atlas for a G -equivariant sheaf \mathcal{F} . Then the conclusions of Theorem 3.12 hold for any G -equivariant coherent sheaf \mathcal{A} on the sheaf stack \mathcal{F} .*

Proof. The existence of a G -equivariant affine atlas for \mathcal{F} gives a G -equivariant surjective affine étale morphism $\rho: V \rightarrow X$, where V is a G -scheme. Since Deligne-Mumford stacks in this paper are assumed to be separated, the proof of Theorem 3.12 applies verbatim to imply the statement. \square

3.4. Equivariant Gysin maps for sheaf stacks in intersection theory. Chang-Li [CL11] also define an intersection-theoretic Gysin map in an analogous fashion, which we denote by the same notation

$$(3.16) \quad 0_{\mathcal{F}}^!: A_*(\mathcal{F}) \longrightarrow A_*(X).$$

Here $A_*(\mathcal{F})$ are the Chow groups of the sheaf stack \mathcal{F} , defined in [CL11], also using local charts (cf. Definition 3.2). This has a natural equivariant lift

$$(3.17) \quad 0_{\mathcal{F}}^{!G}: A_*^G(\mathcal{F}) \longrightarrow A_*^G(X).$$

We remark that we will only be applying the equivariant Gysin map to classes in $A_*^G(\mathcal{F})$ represented by linear combinations of G -invariant integral cycles $Z \subseteq \mathcal{F}$. The general definition is obtained by the same procedure described in Section 2.3, considering the Chow groups

$$A_i^G(\mathcal{F}) := A_{i+l-g}(\mathcal{F} \times^G U)$$

for l -dimensional good pairs (V, U) for the integer $n - i$, where $\mathcal{F} \times^G U$ denotes the descent to $X \times^G U$ of the pullback of \mathcal{F} to the product $X \times U$, and applying the usual Gysin map $0_{\mathcal{F} \times^G U}^!$.

4. ALMOST PERFECT OBSTRUCTION THEORY, VIRTUAL STRUCTURE SHEAF AND VIRTUAL FUNDAMENTAL CYCLE

In this short section, we recall the definitions of an almost perfect obstruction given in [KS20a] and the associated virtual structure sheaf and fundamental cycle and then generalize them to the equivariant setting.

Definition 4.1. (Almost perfect obstruction theory) *Let $X \rightarrow Y$ be a morphism, where X is a Deligne-Mumford stack of finite presentation and Y is a smooth Artin stack of pure dimension. An almost perfect obstruction theory ϕ consists of an étale covering $\{X_\alpha \rightarrow X\}_{\alpha \in A}$ of X and perfect obstruction theories $\phi_\alpha: E_\alpha \rightarrow \mathbb{L}_{X_\alpha/Y}$ of X_α over Y such that the following hold.*

- (1) *For each pair of indices α, β , there exists an isomorphism*

$$\psi_{\alpha\beta}: \mathcal{O}b_{X_\alpha}|_{X_{\alpha\beta}} \longrightarrow \mathcal{O}b_{X_\beta}|_{X_{\alpha\beta}}$$

so that the collection $\{\mathcal{O}b_{X_\alpha} = h^1(E_\alpha^\vee), \psi_{\alpha\beta}\}$ gives descent data of a sheaf $\mathcal{O}b_X$, called the obstruction sheaf, on X .

- (2) *For each pair of indices α, β , there exists an étale covering $\{V_\lambda \rightarrow X_{\alpha\beta}\}_{\lambda \in \Gamma}$ of $X_{\alpha\beta} = X_\alpha \times_X X_\beta$ such that for any λ , the perfect obstruction theories $E_\alpha|_{V_\lambda}$ and $E_\beta|_{V_\lambda}$ are isomorphic and compatible with $\psi_{\alpha\beta}$. This means that there exists an isomorphism*

$$\eta_{\alpha\beta\lambda}: E_\alpha|_{V_\lambda} \longrightarrow E_\beta|_{V_\lambda}$$

in $D^b(\mathrm{Coh}V_\lambda)$ fitting in a commutative diagram

$$(4.1) \quad \begin{array}{ccc} E_\alpha|_{V_\lambda} & \xrightarrow{\eta_{\alpha\beta\lambda}} & E_\beta|_{V_\lambda} \\ \phi_\alpha|_{V_\lambda} \downarrow & & \downarrow \phi_\beta|_{V_\lambda} \\ \mathbb{L}_{X_\alpha/Y}|_{V_\lambda} & \longrightarrow & \mathbb{L}_{X_\beta/Y}|_{V_\lambda} \\ & \searrow & \downarrow \\ & & \mathbb{L}_{V_\lambda/Y} \end{array}$$

which moreover satisfies $h^1(\eta_{\alpha\beta\lambda}^\vee) = \psi_{\alpha\beta}^{-1}|_{V_\lambda}$.

Suppose that the morphism $X \rightarrow Y$ admits an almost perfect obstruction theory. Then the definition implies that the closed embeddings given in diagram (2.4)

$$h^1(\phi_\alpha^\vee): \mathbf{n}_{U_\alpha/Y} \longrightarrow \mathcal{O}b_{X_\alpha}$$

glue to a global closed embedding

$$j_\phi: \mathbf{n}_{X/Y} \longrightarrow \mathcal{O}b_X$$

of sheaf stacks over X . Therefore, the coarse intrinsic normal cone stack $\mathbf{c}_{X/Y}$ embeds as a closed substack into the sheaf stack $\mathcal{O}b_X$.

Definition 4.2. [KS20a] (Virtual structure sheaf) *Let $X \rightarrow Y$ be as above, together with an almost perfect obstruction theory $\phi: X \rightarrow Y$. The virtual structure sheaf of X associated to ϕ is defined as*

$$[\mathcal{O}_X^{\mathrm{vir}}] := 0_{\mathcal{O}b_X}^![\mathcal{O}_{\mathbf{c}_{X/Y}}] \in K_0(X).$$

It is not hard to show that an almost perfect obstruction theory satisfies the axioms of a semi-perfect obstruction theory and thus by [CL11] induces a virtual fundamental cycle $[X]^{\mathrm{vir}} \in A_*(X)$. This can be expressed directly using the coarse intrinsic normal cone stack $\mathbf{c}_{X/Y}$.

Definition 4.3. [CL11, KS20a] (Virtual fundamental cycle) *Let $X \rightarrow Y$ be as above, together with an almost perfect obstruction theory $\phi: X \rightarrow Y$. The virtual fundamental cycle of X associated to ϕ is defined as*

$$[X]^{\mathrm{vir}} := 0_{\mathcal{O}b_X}^![\mathbf{c}_{X/Y}] \in A_*(X),$$

where $0_{\mathcal{O}b_X}^!$ denotes the Gysin map (3.16) and $[\mathbf{c}_{X/Y}] \in A_*(\mathcal{O}b_X)$ is the cycle associated to the closed substack $\mathbf{c}_{X/Y}$ of $\mathcal{O}b_X$ (cf. [CL11]).

The above definition and constructions admit direct analogues in the equivariant context. We now briefly explain the necessary adjustments.

As usual, G denotes a linear algebraic group, X is a Deligne-Mumford stack with an action of G , Y is a smooth Artin stack of pure dimension with a G -action, and $f: X \rightarrow Y$ a G -equivariant morphism.

When $G = \mathbb{C}^*$ and Y is a point, the notion of G -equivariant almost perfect obstruction theory was defined in [KS20b, Definition 5.1] in order to develop a virtual torus localization formula. It is clear how to generalize the definition of an almost perfect obstruction theory to our setting as follows.

Definition 4.4. (*G*-equivariant almost perfect obstruction theory) *Let $f: X \rightarrow Y$ be a G -equivariant morphism as above. A G -equivariant almost perfect obstruction theory ϕ consists of the following data:*

- (a) *A G -equivariant étale covering $\{X_\alpha \rightarrow X\}_{\alpha \in A}$ of X .*
- (b) *For each index $\alpha \in A$, an object $E_\alpha \in D^G(X_\alpha)$ and a morphism $\phi_\alpha: E_\alpha \rightarrow \mathbb{L}_{X_\alpha}$ in $D^G(X_\alpha)$ which gives a G -equivariant perfect obstruction theory on X_α .*

These are required to satisfy the following conditions:

- (1) *For each pair of indices α, β , there exists a G -equivariant isomorphism*

$$\psi_{\alpha\beta}: \mathcal{O}b_{X_\alpha}|_{X_{\alpha\beta}} \longrightarrow \mathcal{O}b_{X_\beta}|_{X_{\alpha\beta}}$$

so that the collection $\{\mathcal{O}b_{X_\alpha} = h^1(E_\alpha^\vee), \psi_{\alpha\beta}\}$ gives descent data of a G -equivariant coherent sheaf $\mathcal{O}b_X$, called the obstruction sheaf, on X .

- (2) *For each pair of indices α, β , there exists a G -equivariant étale covering $\{V_\lambda \rightarrow X_{\alpha\beta}\}_{\lambda \in \Gamma}$ of $X_{\alpha\beta} = X_\alpha \times_X X_\beta$ such that for any λ , the perfect obstruction theories $\phi_\alpha|_{V_\lambda}$ and $\phi_\beta|_{V_\lambda}$ are isomorphic and compatible with $\psi_{\alpha\beta}$. This means that there exists an isomorphism*

$$\eta_{\alpha\beta\lambda}: E_\alpha|_{V_\lambda} \longrightarrow E_\beta|_{V_\lambda}$$

in $D^G(V_\lambda)$ fitting in a commutative diagram

$$(4.2) \quad \begin{array}{ccc} E_\alpha|_{V_\lambda} & \xrightarrow{\eta_{\alpha\beta\lambda}} & E_\beta|_{V_\lambda} \\ \phi_\alpha|_{V_\lambda} \downarrow & & \downarrow \phi_\beta|_{V_\lambda} \\ \mathbb{L}_{X_\alpha}|_{V_\lambda} & \longrightarrow & \mathbb{L}_{X_\beta}|_{V_\lambda} \\ & \searrow & \downarrow \\ & & \mathbb{L}_{V_\lambda} \end{array}$$

which moreover satisfies $h^1(\eta_{\alpha\beta\lambda}^\vee) = \psi_{\alpha\beta}^{-1}|_{V_\lambda}$.

In the above, $D^G(X_\alpha)$ and $D^G(V_\lambda)$ denote the bounded derived categories of T -equivariant quasi-coherent sheaves on U_α and V_λ respectively.

By definition, the obstruction sheaf $\mathcal{O}b_X$ is G -equivariant and the closed embedding $j_\phi: \mathfrak{n}_{X/Y} \rightarrow \mathcal{O}b_X$ is G -invariant. Hence, $\mathfrak{c}_{X/Y}$ gives a G -equivariant sheaf $\mathcal{O}_{\mathfrak{c}_{X/Y}}$ on the sheaf stack $\mathcal{O}b_X$ and a G -invariant cycle $[\mathfrak{c}_{X/Y}] \in A_*^G(\mathcal{O}b_X)$. We may thus give the following definition using the equivariant Gysin maps (3.11) and (3.17).

Definition 4.5. *The G -equivariant virtual structure sheaf and virtual fundamental cycle are defined as*

$$\begin{aligned} [\mathcal{O}_X^{\text{vir}}] &:= 0_{\mathcal{O}b_X}^{!,G} [\mathcal{O}_{\mathfrak{c}_{X/Y}}] \in K_0^G(X), \\ [X]^{\text{vir}} &:= 0_{\mathcal{O}b_X}^{!,G} [\mathfrak{c}_{X/Y}] \in A_*^G(X). \end{aligned}$$

5. VIRTUAL RIEMANN-ROCH FOR ALMOST PERFECT OBSTRUCTION THEORIES

In this section, we prove the main results of the paper, which are virtual Riemann-Roch theorems for morphisms equipped with an almost perfect obstruction theory in the non-equivariant and equivariant cases, which remove several technical assumptions and generalize the virtual Riemann-Roch theorems of Fantechi-Göttsche [FG10] and Ravi-Sreedhar [RS20]. In particular, we show that a virtual Todd class is always defined and takes the expected form in the presence of an appropriate resolution of the almost perfect obstruction theory.

Despite the fact that the usual Riemann-Roch theorem is the special case of the equivariant Riemann-Roch theorem when the group is trivial, it will be useful to develop the non-equivariant case first, as it is of independent interest and will be the basis used to establish the theorem in the equivariant case.

5.1. The non-equivariant case. Let $f: X \rightarrow Y$ be a morphism from an algebraic space X to a smooth Artin stack Y of pure dimension equipped with an almost perfect obstruction theory ϕ inducing a closed embedding $j_\phi: \mathfrak{n}_{X/Y} \rightarrow \mathcal{O}b_X$.

Definition 5.1. *The virtual Todd class of X over Y is the class*

$$(5.1) \quad \mathrm{td}^{\mathrm{vir}}(X/Y) := \tau_X([\mathcal{O}_X^{\mathrm{vir}}]) \in A_*(X),$$

where τ_X is the Riemann-Roch transformation (2.1) of X .

Observe that the virtual Todd class only depends on the embedding j_ϕ . The significance of the definition is in that it gives the natural substitute in the setting of almost perfect obstruction theory for the expression $\mathrm{td}(T_{X/Y}^{\mathrm{vir}}) \cap [X]^{\mathrm{vir}}$ that appears in the virtual Riemann-Roch theorem, even though the K -theoretic virtual tangent bundle $T_{X/Y}^{\mathrm{vir}}$ does not make sense.

As a consequence of the properties of the Riemann-Roch transformation τ_X , we obtain a virtual Riemann-Roch formula as follows.

Theorem 5.2. *When f is proper and Y is a smooth scheme, we have for any $V \in K^0(X)$*

$$\mathrm{ch}(f_*(V \otimes [\mathcal{O}_X^{\mathrm{vir}}])) \cdot \mathrm{td}(T_Y) \cap [Y] = f_*(\mathrm{ch}(V) \cdot \mathrm{td}^{\mathrm{vir}}(X/Y)).$$

Proof. The proof is formal using the properties of τ_X identically as in [FG10, Lemma 3.5]. \square

In the rest of this subsection, we explore the virtual Todd class in more detail. Our motivation is to provide an explicit formula in terms of vector bundles on X .

In the case that the embedding j_ϕ is induced by a perfect obstruction theory ψ , then $\mathrm{td}^{\mathrm{vir}}(X/Y)$ recovers the usual Riemann-Roch term, as shown by the next proposition. ψ does not need to be related to ϕ in any other way beyond inducing the embedding j_ϕ .

Proposition 5.3. *Suppose that $\psi: E \rightarrow \mathbb{L}_{X/Y}$ is a perfect obstruction theory on $f: X \rightarrow Y$ with obstruction sheaf $\mathcal{O}b_X$ such that $j_\psi = h^1(\psi^\vee) = j_\phi$*

and $E = [E^{-1} \rightarrow E^0]$ is a global resolution by a complex of locally free sheaves. Then

$$\mathrm{td}^{\mathrm{vir}}(X/Y) = \frac{\mathrm{td}(E_0)}{\mathrm{td}(E_1)} \mathrm{td}(f^*T_Y) \cap [X]^{\mathrm{vir}} = \mathrm{td}(E^\vee) \mathrm{td}(f^*T_Y) \cap [X]^{\mathrm{vir}},$$

where $E_0 = (E^0)^\vee, E_1 = (E^{-1})^\vee$.

Proof. We first remark that the virtual cycles and virtual structure sheaves induced by the almost perfect obstruction theory ϕ and the perfect obstruction theory ψ are equal, since by definition they are both given by the expression $0_{E_1}^1[C_1] \in A_*(X)$ and $0_{E_1}^1[\mathcal{O}_{C_1}] \in K_0(X)$, where $C_1 = \mathfrak{c}_{X/Y} \times_{\mathcal{O}_X} E_1$. We may thus write $[X]^{\mathrm{vir}}$ and $[\mathcal{O}_X^{\mathrm{vir}}]$ unambiguously.

Writing $T_{X/Y}^{\mathrm{vir}} = [E_0] - [E_1] + [f^*T_Y]$, we have by applying Theorem 2.8 with G being the trivial group that

$$\mathrm{td}^{\mathrm{vir}}(X/Y) = \tau_X([\mathcal{O}_X^{\mathrm{vir}}]) = \mathrm{td}(T_{X/Y}^{\mathrm{vir}}) \cap [X]^{\mathrm{vir}} = \mathrm{td}(E^\vee) \mathrm{td}(f^*T_Y) \cap [X]^{\mathrm{vir}}.$$

□

More generally, suppose that the K -theory class $[T_{X/Y}] - [\mathcal{O}_X] \in K_0(X)$ is contained in the image of the natural map

$$\kappa: K^0(X) \longrightarrow K_0(X)$$

so that $[T_{X/Y}] - [\mathcal{O}_X] = \kappa(F)$ for some $F \in K^0(X)$.

We can then formulate the following conjecture.

Conjecture 5.4. $\mathrm{td}^{\mathrm{vir}}(X/Y) = \mathrm{td}(F) \cap [X]^{\mathrm{vir}}$ for any $F \in K^0(X)$ satisfying $\kappa(F) = [T_{X/Y}] - [\mathcal{O}_X]$.

Proposition 5.3 can be rephrased as saying that the conjecture holds for $F = E^\vee = [E_0] - [E_1]$ when $E = [E^{-1} \rightarrow E^0]$ is a perfect obstruction theory compatible with the embedding j_ϕ .

We now show that the conjecture is true under certain assumptions which give a presentation of the K -theory class $[T_{X/Y}] - [\mathcal{O}_X] = [E_0] - [E_1]$ as an element of $K^0(X)$, but are weaker than the existence of a perfect obstruction theory inducing the closed embedding j_ϕ .

Definition 5.5. Let $E = [E^{-1} \xrightarrow{d} E^0]$ be a complex of locally free sheaves on X satisfying:

- (1) $h^0(E) = \Omega_{X/Y}$ and $h^1(E^\vee) = \mathcal{O}_X$;
- (2) there exists a morphism $E^0 \rightarrow \mathbb{L}_{X/Y}$ such that the natural morphism $E^0 \rightarrow h^0(E) = \Omega_{X/Y}$ factors through $\mathbb{L}_{X/Y} \rightarrow h^0(\mathbb{L}_{X/Y}) = \Omega_{X/Y}$.

The complex $E^\vee = [E_0 \rightarrow E_1]$ is called a global presentation of the virtual tangent bundle of f . We then set $T_{X/Y}^{\mathrm{vir}} = [E_0] - [E_1] + [f^*(T_Y)] \in K^0(X)$.

It is clear that $\kappa(T_{X/Y}^{\mathrm{vir}}) = [T_{X/Y}] - [\mathcal{O}_X] + [f^*(T_Y)] \in K_0(X)$ for any global presentation of the virtual tangent bundle of ϕ .

Theorem 5.6. Suppose that X admits a closed embedding into a smooth algebraic space and let $E^\vee = [E_0 \rightarrow E_1]$ be a global presentation of the virtual tangent bundle of f . Then

$$\mathrm{td}^{\mathrm{vir}}(X/Y) = \frac{\mathrm{td}(E_0)}{\mathrm{td}(E_1)} \mathrm{td}(f^*T_Y) \cap [X]^{\mathrm{vir}} = \mathrm{td}(T_{X/Y}^{\mathrm{vir}}) \cap [X]^{\mathrm{vir}}.$$

Before proving the theorem, we need the following auxiliary lemma.

Lemma 5.7. *Let $E^\vee = [E_0 \rightarrow E_1]$ be a global presentation of the virtual tangent bundle of f . Then there exists an étale cover $\{X_\alpha \rightarrow X\}$ and perfect obstruction theories $\psi_\alpha: E|_{X_\alpha} \rightarrow \mathbb{L}_{X_\alpha}$ such that $h^1(\psi_\alpha^\vee) = j_\phi|_{X_\alpha}$.*

Proof. By condition (2) in Definition 5.5, we have a presentation

$$\mathbb{L}_{X/Y} = [\mathcal{F}^{-1} \rightarrow E^0]$$

for some coherent sheaf \mathcal{F}^{-1} on X .

By the existence of ϕ , there exists an étale cover $\{X_\alpha \rightarrow X\}$ and perfect obstruction theories $\phi_\alpha: E_\alpha \rightarrow \mathbb{L}_{X_\alpha}$ such that $h^1(\phi_\alpha^\vee) = j_\phi|_{X_\alpha}$. It thus suffices (up to possibly shrinking X_α) to construct isomorphisms $E|_{X_\alpha} \cong E_\alpha$ inducing the identity map on h^1 .

Shrinking X_α around a point $x \in X_\alpha$, we may assume that X_α is affine, $E_\alpha = [G^{-1} \xrightarrow{h} G^0]$ is a complex of locally free sheaves which is minimal at x , meaning that $h|_x = 0$, and ϕ_α is given by a morphism of complexes

$$\begin{array}{ccc} G^{-1} & \longrightarrow & G^0 \\ \downarrow & & \downarrow \\ \mathcal{F}^{-1} & \longrightarrow & E^0. \end{array}$$

$G_1 = (G^{-1})^\vee$ surjects onto $h^1(E_\alpha^\vee) = \mathcal{O}b_X|_{X_\alpha}$ and the same is true for $E_1|_{X_\alpha}$, so we obtain a morphism $f: E_1|_{X_\alpha} \rightarrow G_1$ fitting into a commutative diagram

$$(5.2) \quad \begin{array}{ccccc} E_0|_{X_\alpha} & \longrightarrow & E_1|_{X_\alpha} & \longrightarrow & \mathcal{O}b_X|_{X_\alpha} \\ & & \downarrow f & \nearrow & \\ G_0 & \xrightarrow{h} & G_1 & & \end{array}$$

Since E_α is minimal at x , we have that $f|_x: E_1|_x \rightarrow G_1|_x \cong \mathcal{O}b_X|_x$ is surjective, so, up to further possible shrinking, we may assume that f is surjective and we can split $E_1|_{X_\alpha} = G_1 \oplus K_1$ such that f is the projection onto the first factor.

By the exactness of the top row in (5.2) at its middle term, the composition $E_0|_{X_\alpha} \rightarrow G_1 \oplus K_1 \xrightarrow{\pi_2} K_1$ is surjective and thus we may also split $E_0|_{X_\alpha} = G'_0 \oplus K_1$ to obtain

$$(5.3) \quad \begin{array}{ccccc} H_0 \oplus K_1 & \xrightarrow{d} & G_1 \oplus K_1 & \longrightarrow & \mathcal{O}b_X|_{X_\alpha} \\ & & \downarrow \pi_1 & \nearrow & \\ G_0 & \xrightarrow{h} & G_1 & & \end{array}$$

where $d = d' \oplus \text{id}_{K_1}$.

Dualizing, we get

$$(5.4) \quad \begin{array}{ccc} G^{-1} \oplus K^{-1} & \longrightarrow & H^0 \oplus K^{-1} \\ \uparrow \text{inc}_1 & & \\ G^{-1} & \longrightarrow & G^0. \end{array}$$

The composition $K^{-1} \xrightarrow{\text{inc}_2} H^0 \oplus K^{-1} \rightarrow \Omega_{X/Y}|_{X_\alpha}$ is then zero and thus we must have a surjection $H^0 \rightarrow \Omega_{X/Y}|_{X_\alpha}$. The same is true for $G^0 \rightarrow \Omega_{X/Y}|_{X_\alpha}$ and hence there exists a morphism $g: H^0 \rightarrow G^0$ fitting in a commutative diagram

$$(5.5) \quad \begin{array}{ccc} H^0 & \longrightarrow & \Omega_{X/Y}|_{X_\alpha} \\ g \downarrow & \nearrow & \\ G^0 & & \end{array}$$

By minimality, $G^0|_x \cong \Omega_{X/Y}|_x$ and hence we may assume that g is a surjection. But H^0 and G^0 have the same rank since $H^0|_x$ and $G^0|_x$ have equal dimensions, so g must be an isomorphism around x .

It is now clear that there exists an isomorphism $k: G^0 \rightarrow H^0$ fitting into the commutative diagram (5.4) as follows:

$$(5.6) \quad \begin{array}{ccc} G^{-1} \oplus K^{-1} & \longrightarrow & H^0 \oplus K^{-1} \\ \uparrow \text{inc}_1 & & k \oplus 0 \uparrow \\ G^{-1} & \longrightarrow & G^0. \end{array}$$

This gives the required isomorphism of complexes and we are done. \square

Remark 5.8. *While we won't explicitly use this, the lemma actually proves that the perfect obstruction theories ϕ_α forming part of the data of ϕ can, up to possible shrinking, be written in the form $E|_{X_\alpha} \rightarrow \mathbb{L}_{X_\alpha}$.*

We can now proceed to prove Theorem 5.6.

Proof of Theorem 5.6. By the definition of E , we have a presentation $\mathbb{L}_{X/Y} = [\mathcal{F}^{-1} \rightarrow E^0]$ for some coherent sheaf \mathcal{F}^{-1} on X . Write $\mathcal{F}_1 = (\mathcal{F}^{-1})^\vee$, so that

$$\mathfrak{n}_{X/Y} = \text{coker}(E_0 \rightarrow \mathcal{F}_1) = h^1(\mathbb{T}_{X/Y}).$$

We then have the surjection $r: E_1 \rightarrow h^1(E^\vee) = \mathcal{O}b_X$, which factors as a composition $E_1 \rightarrow [E_1/E_0] \rightarrow \mathcal{O}b_X$ and fits into a diagram with Cartesian squares

$$(5.7) \quad \begin{array}{ccccc} C_1 & \longrightarrow & M_1 & \xrightarrow{\iota} & E_1 \\ \downarrow & & \downarrow & & \downarrow r \\ \mathfrak{c}_{X/Y} & \longrightarrow & \mathfrak{n}_{X/Y} & \xrightarrow{j_\phi} & \mathcal{O}b_X. \end{array}$$

Write $N_1 = \text{Spec}_{\mathcal{O}_X}(\text{Sym } \mathcal{F}^{-1})$. We claim that $M_1 = N_1$.

By Lemma 5.7, we have an étale cover $\{X_\alpha \rightarrow X\}$ and perfect obstruction theories $\psi_\alpha: E|_{X_\alpha} \rightarrow \mathbb{L}_{X_\alpha}$ such that $h^1(\psi_\alpha^\vee) = j_\phi|_{X_\alpha}$.

Restricting to each X_α , the existence of ψ_α locally enhances diagram (5.7) to a diagram with Cartesian squares

$$\begin{array}{ccccccc}
E_0|_{X_\alpha} & \longrightarrow & C_1|_{X_\alpha} & \longrightarrow & N_1|_{X_\alpha} & \xrightarrow{\iota_\alpha} & E_1|_{X_\alpha} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_\alpha & \xrightarrow{0_{\mathcal{C}_{X/Y}|_{X_\alpha}}} & \mathcal{C}_{X/Y}|_{X_\alpha} & \longrightarrow & \mathcal{N}_{X/Y}|_{X_\alpha} & \xrightarrow{h^1/h^0(\psi_\alpha^\vee)} & [E_1|_{X_\alpha}/E_0|_{X_\alpha}] \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \mathfrak{c}_{X/Y}|_{X_\alpha} & \longrightarrow & \mathfrak{n}_{X/Y}|_{X_\alpha} & \xrightarrow{h^1(\psi_\alpha^\vee)=j_\phi|_{X_\alpha}} & \mathcal{O}_{b_X}|_{X_\alpha},
\end{array}$$

which immediately implies that the closed embeddings $\iota_\alpha: N_1|_{X_\alpha} \rightarrow E_1|_{X_\alpha}$ glue to a closed embedding $N_1 \rightarrow E_1$ giving the desired identification $M_1 = N_1$ as substacks of E_1 .

The distinguished triangle

$$f^*\Omega_Y \longrightarrow \mathbb{L}_X \longrightarrow \mathbb{L}_{X/Y} \longrightarrow f^*\Omega_Y[1]$$

gives rise to a diagram with Cartesian squares

$$\begin{array}{ccccccc}
(5.8) & E_0 & \longrightarrow & F_0 & \longrightarrow & C_1 & \longrightarrow & N_1 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& X & \longrightarrow & f^*T_Y & \longrightarrow & \mathcal{C}_{X/Y} & \longrightarrow & \mathcal{N}_{X/Y} \\
& & & \downarrow & & \downarrow & & \downarrow \\
& & & X & \longrightarrow & \mathcal{C}_X & \longrightarrow & \mathcal{N}_X
\end{array}$$

with smooth vertical arrows, where $F_0 = (F^0)^\vee$ and

$$(5.9) \quad 0 \longrightarrow E_0 \longrightarrow F_0 \longrightarrow f^*T_Y \longrightarrow 0$$

is an exact sequence of vector bundles.

We now have a diagram of distinguished triangles

$$\begin{array}{ccccccc}
f^*\Omega_Y & \longrightarrow & [\mathcal{F}^{-1} \rightarrow F^0] & \longrightarrow & [\mathcal{F}^{-1} \rightarrow E^0] & \longrightarrow & f^*\Omega_Y[1] \\
\downarrow & & & & \downarrow & & \downarrow \\
f^*\Omega_Y & \longrightarrow & \mathbb{L}_X & \longrightarrow & \mathbb{L}_{X/Y} & \longrightarrow & f^*\Omega_Y[1].
\end{array}$$

Since $[\mathcal{F}^{-1} \rightarrow E^0]$ is a presentation of $\mathbb{L}_{X/Y}$, the vertical arrows are isomorphisms and we obtain an induced (non-canonical) isomorphism $\mathbb{L}_X \cong [\mathcal{F}^{-1} \rightarrow F^0]$.

Thus, writing $\pi_1: C_1 \rightarrow X$ for the projection and using the assumption that X admits a closed embedding into a smooth algebraic space, [FG10, Proposition 3.1], (5.8) and (5.9) imply that

$$(5.10) \quad \tau_{C_1}(\mathcal{O}_{C_1}) = \pi_1^*(\mathrm{td}(F_0)) \cap [C_1] = \pi_1^*(\mathrm{td}(E_0))\pi_1^*(\mathrm{td}(f^*T_Y)) \cap [C_1].$$

By definition, the virtual structure sheaf of X is equal to

$$(5.11) \quad [\mathcal{O}_X^{\mathrm{vir}}] = 0_{\mathcal{O}_{b_X}}^![\mathcal{O}_{\mathfrak{c}_{X/Y}}] = 0_{E_1}^![\mathcal{O}_{C_1}].$$

Using the properties of the Riemann-Roch transformation, (5.11) gives

$$(5.12) \quad \tau_X([\mathcal{O}_X^{\text{vir}}]) = \tau_X(0_{E_1}^![\mathcal{O}_{C_1}]) = \text{td}(-E_1) \cdot 0_{E_1}^!(\tau_{E_1}([\mathcal{O}_{C_1}])).$$

Denoting the embedding $C_1 \rightarrow E_1$ by k and the projection $p_1: E_1 \rightarrow X$, using the projection formula and $\pi_1 = p_1 \circ k$, we also have by (5.10)

$$(5.13) \quad \begin{aligned} \tau_{E_1}([\mathcal{O}_{C_1}]) &= k_*\tau_{C_1}([\mathcal{O}_{C_1}]) = k_*(\pi_1^*(\text{td}(F_0)) \cap [C_1]) = \\ &= k_*(k^*p_1^*(\text{td}(F_0)) \cap [C_1]) = \\ &= p_1^*(\text{td}(F_0)) \cap k_*[C_1] = \\ &= p_1^*(\text{td}(E_0))p_1^*(\text{td}(f^*T_Y)) \cap k_*[C_1]. \end{aligned}$$

Combining (5.12) and (5.13) finally yields

$$\begin{aligned} \text{td}^{\text{vir}}(X/Y) &= \tau_X([\mathcal{O}_X^{\text{vir}}]) = \text{td}(-E_1) \cdot 0_{E_1}^!(p_1^*(\text{td}(F_0)) \cap k_*[C_1]) = \\ &= \text{td}(-E_1) \cdot \text{td}(F_0) \cap [X]^{\text{vir}} = \frac{\text{td}(E_0)}{\text{td}(E_1)} \text{td}(f^*T_Y) \cap [X]^{\text{vir}}, \end{aligned}$$

since $0_{E_1}^!(k_*[C_1]) = [X]^{\text{vir}}$ and $0_{E_1}^! \circ p_1^* = \text{id}$, concluding the proof. \square

5.2. The equivariant case. Let now $f: X \rightarrow Y$ be a G -equivariant morphism between an algebraic space X with G -action and a smooth, pure-dimensional G -scheme Y and ϕ be a G -equivariant almost perfect obstruction theory on f , inducing a closed embedding $j_\phi: \mathbf{c}_{X/Y} \rightarrow \mathcal{O}b_X$.

By Definition 4.5, $[\mathcal{O}_X^{\text{vir}}]$ is then naturally an element of $K_0^G(X)$.

Definition 5.9. *The equivariant virtual Todd class of X over Y is the class*

$$(5.14) \quad \text{td}^{\text{vir},G}(X/Y) := \tau_X^G([\mathcal{O}_X^{\text{vir}}]) \in A_*^G(X),$$

where τ_X^G is the equivariant Riemann-Roch transformation of X .

As before, we deduce a tautological equivariant virtual Riemann-Roch formula from the definition.

Theorem 5.10. *When f is proper, we have for any $V \in K_G^0(X)$*

$$\text{ch}(f_*(V \otimes [\mathcal{O}_X^{\text{vir}}])) \cdot \text{td}(T_Y) \cap [Y] = f_*(\text{ch}(V) \cdot \text{td}^{\text{vir}}(X/Y)).$$

We would now like to generalize Theorem 5.6 to the present equivariant context. We will achieve this by reducing to the non-equivariant case as follows. Let (V, U) be an l -dimensional good pair for the integer $n - i$. Recall that τ_X^G is defined componentwise by the diagram (2.3)

$$\begin{array}{ccccc} K_0^G(X \times V) & \xrightarrow{j^*} & K_0^G(X \times U) & \longrightarrow & K_0(X \times^G U) \\ \pi_X^* \uparrow & \nearrow \pi_X^* & & \nearrow s_U & \downarrow \left(\frac{\tau_{X \times^G U}}{\text{td}(X \times^G (U \times V))} \right)_{i+l-g} \\ K_0^G(X) & & \xrightarrow{(\tau_X^G)_i} & & A_{i+l-g}(X \times^G U). \end{array}$$

Write π_V for both the vector bundle projections $X \times V \rightarrow V$ and $X \times^G (U \times V) \rightarrow X \times^G U$. By abuse of notation, we use the letter V to denote the latter vector bundle on $X \times^G U$.

We also write $s_U(\mathcal{F})$ for the sheaf (or complex of sheaves) on $X \times^G U$ induced by pulling back a G -equivariant sheaf \mathcal{F} (or complex of sheaves)

on X via π_X and descending to $X \times^G U$. Similarly, we can define $s_U(B) \in A_*(X \times^G U)$ for any class $B \in A_*^G(X)$.

We may work with $X \times^G U$ by the following proposition.

Proposition 5.11. *A G -equivariant almost perfect obstruction theory ϕ on $f: X \rightarrow Y$ induces a canonically defined almost perfect obstruction theory ϕ_U on the morphism $f_U: X \times^G U \rightarrow Y \times^G U$.*

Their virtual structure sheaves and virtual fundamental cycles satisfy

$$(5.15) \quad \begin{aligned} s_U([\mathcal{O}_X^{\text{vir}}]) &= [\mathcal{O}_{X \times^G U}^{\text{vir}}] \in K_0(X \times^G U), \\ s_U([X]^{\text{vir}}) &= [X \times^G U]^{\text{vir}} \in A_*(X \times^G U). \end{aligned}$$

Proof. For any index α , pulling back the G -equivariant perfect obstruction theory $\phi_\alpha: E_\alpha \rightarrow \mathbb{L}_{X_\alpha/Y}$ via the projection $\pi_X: X \times U \rightarrow X$ gives a morphism

$$(5.16) \quad \pi_X^* E_\alpha \longrightarrow \pi_X^* \mathbb{L}_{X_\alpha/Y}.$$

Since $\mathbb{L}_{X \times U/Y \times U} = \pi_X^* \mathbb{L}_{X/Y}$ descends to $\mathbb{L}_{X \times^G U/Y \times^G U}$ on $X \times^G U$, (5.16) descends to give a perfect obstruction theory

$$(5.17) \quad (\phi_U)_\alpha: (E_U)_\alpha := s_U(E_\alpha) \longrightarrow \mathbb{L}_{X_\alpha \times^G U/Y \times^G U}.$$

It is a formal exercise to verify that these morphisms form part of the data of an almost perfect obstruction theory ϕ_U on $X \times^G U$ by pulling back the isomorphisms $\psi_{\alpha\beta}$ and $\eta_{\alpha\beta\lambda}$ in Definition 4.4 via π_X and then descending to $X \times^G U$. The obstruction sheaf of ϕ_U is $s_U(\mathcal{O}_{b_X})$.

We have that $\mathfrak{c}_{X/Y} \times_X (X \times U) = \mathfrak{c}_{X \times U/Y \times U}$. The equality (5.15) then follows immediately from the definition of the Gysin map $0_{\mathcal{O}_{b_X}}^{!G}$ in Theorem 3.12 using an affine G -equivariant étale atlas $V \rightarrow X \times U$, the definition of $0_{s_U(\mathcal{O}_{b_X})}^!$ and (5.17). \square

A G -equivariant global presentation of the virtual tangent bundle of f can be defined in analogy with Definition 5.5.

Definition 5.12. *A G -equivariant global presentation of the virtual tangent bundle of f is a global presentation in the sense of Definition 5.5 such that the complex E and the morphism $E^0 \rightarrow \mathbb{L}_X$ are G -equivariant.*

Proposition 5.13. *A G -equivariant global presentation $E^\vee = [E_0 \rightarrow E_1]$ of the virtual tangent bundle of f induces a global presentation $F^\vee = [F_0 \rightarrow F_1]$ of the virtual tangent bundle of f_U , where $F_i = s_U(E_i)$.*

Proof. This is immediate from Proposition 5.11 and the definitions. \square

Theorem 5.14. *Suppose that X admits a G -invariant closed embedding into a smooth algebraic space with a G -action and let $E^\vee = [E_0 \rightarrow E_1]$ be a G -equivariant global presentation of the virtual tangent bundle of f . Then*

$$\text{td}^{\text{vir},G}(X/Y) = \frac{\text{td}^G(E_0)}{\text{td}^G(E_1)} \cdot \frac{\text{td}^G(f^*T_Y)}{\text{td}^G(\mathfrak{g})} \cap [X]^{\text{vir}} = \text{td}^G(T_{X/Y}^{\text{vir}}) \cap [X]^{\text{vir}},$$

where $T_{X/Y}^{\text{vir}} = [E_0] - [E_1] + [f^*T_Y] - [\mathfrak{g}] \in K_G^0(X)$ and \mathfrak{g} is the adjoint representation of G .

Proof. A G -invariant closed embedding of X into a smooth algebraic space with a G -action induces a closed embedding of any $X \times^G U$ into a smooth algebraic space for any good pair (V, U) . By Propositions 5.11, 5.13 and Theorem 5.6, we have for the almost perfect obstruction theory ϕ_U

$$(5.18) \quad \tau_{X \times^G U}([\mathcal{O}_{X \times^G U}^{\text{vir}}]) = \frac{\text{td}(F_0)}{\text{td}(F_1)} \text{td}(f_U^* T_{Y \times^G U}) \cap [X \times^G U]^{\text{vir}},$$

and hence, using (5.15),

$$(5.19) \quad \begin{aligned} (\tau_X^G)_i([\mathcal{O}_X^{\text{vir}}]) &= \left(\frac{\tau_{X \times^G U}}{\text{td}(X \times^G (U \times V))} \right)_{i+l-g} (s_U([\mathcal{O}_X^{\text{vir}}])) \\ &= \left(\frac{\tau_{X \times^G U}}{\text{td}(X \times^G (U \times V))} \right)_{i+l-g} ([\mathcal{O}_{X \times^G U}^{\text{vir}}]). \end{aligned}$$

Since $F_i = s_U(E_i)$, for any $B \in A_*^G(X)$ we have

$$(5.20) \quad \text{td}(F_i) \cap s_U(B) = \text{td}^G(E_i) \cap B.$$

Moreover, $T_{Y \times^G U}$ is the descent of the equivariant perfect complex of amplitude $[-1, 0]$

$$\mathfrak{g} \longrightarrow \pi_Y^* T_Y \oplus \pi_U^* T_U$$

on $Y \times U$. Thus, since T_U is equivariantly isomorphic to the vector bundle V over U , we obtain

$$(5.21) \quad \text{td}(f_U^* T_{Y \times^G U}) = \frac{\text{td}^G(f^* T_Y)}{\text{td}^G(f^* \mathfrak{g})} \cdot \text{td}^G(V).$$

Combining (5.18), (5.19), (5.20) and (5.21), we get

$$(\tau_X^G)_i([\mathcal{O}_X^{\text{vir}}]) = \left(\frac{\text{td}^G(E_0)}{\text{td}^G(E_1)} \cdot \frac{\text{td}^G(f^* T_Y)}{\text{td}^G(\mathfrak{g})} \cap [X]^{\text{vir}} \right)_i$$

for any i , which concludes the proof. \square

6. COSECTION LOCALIZATION

In this last section, we examine the compatibility of the Riemann-Roch transformation, at the levels of generality considered earlier in the paper, with localization by cosection [KL13, KL17, KS20b], a standard tool in the treatment of virtual cycles and virtual structure sheaves. Namely, we prove cosection localized versions of the Riemann-Roch theorems established in the preceding section.

To fix notation, let $f: X \rightarrow Y$ be a G -equivariant morphism from an algebraic space X with G -action to a smooth, pure-dimensional G -scheme stack Y and suppose that f is equipped with a G -equivariant almost perfect obstruction theory ϕ . We also assume given a G -invariant cosection $\sigma: \mathcal{O}_b X \rightarrow \mathcal{O}_X$ and write $X(\sigma)$ for its vanishing locus.

As usual, we begin with the non-equivariant case where the group G is trivial.

By [KL13, KL17, KS20b], the existence of the cosection σ implies that the virtual cycle $[X]^{\text{vir}} \in A_*(X)$ and virtual structure sheaf $[\mathcal{O}_X^{\text{vir}}] \in K_0(X)$ can be naturally localized to the locus $X(\sigma)$. We thus obtain their cosection localized counterparts $[X]_{\text{loc}}^{\text{vir}} \in A_*(X(\sigma))$ and $[\mathcal{O}_{X, \text{loc}}^{\text{vir}}] \in K_0(X(\sigma))$.

The following theorem is a cosection localized Riemann-Roch theorem.

Theorem 6.1. *Suppose that X admits a closed embedding into a smooth algebraic space and let $E^\vee = [E_0 \rightarrow E_1]$ be a global presentation of the virtual tangent bundle of f . Then*

$$\tau_{X(\sigma)}([\mathcal{O}_{X,\text{loc}}^{\text{vir}}]) = \text{td}(E^\vee) \text{td}(f^*T_Y) \cap [X]_{\text{loc}}^{\text{vir}}.$$

Proof. The proof follows verbatim the notation and steps of the proof of Theorem 5.6. We explain the necessary modifications.

(5.10) still holds and thus we have

$$\tau_{C_1}(\mathcal{O}_{C_1}) = \pi_1^*(\text{td}(E_0))\pi_1^*(\text{td}(f^*T_Y)) \cap [C_1].$$

Let $E_1(\sigma) = E_1|_{X(\sigma)} \cup \ker(\sigma|_{U(\sigma)})$ considered as a closed algebraic subspace of the vector bundle E_1 . The reduced support of C_1 lies in $E_1(\sigma)$ and, by definition, the cosection localized virtual cycle and virtual structure sheaf of X are defined to be

$$\begin{aligned} [X]_{\text{loc}}^{\text{vir}} &= 0_{\mathcal{O}_{b_X,\sigma}}^! [\mathfrak{c}_{X/Y}] = 0_{E_1,\sigma}^! [C_1] \in A_*(X(\sigma)), \\ [\mathcal{O}_{X,\text{loc}}^{\text{vir}}] &= 0_{\mathcal{O}_{b_X,\sigma}}^! [\mathcal{O}_{\mathfrak{c}_{X/Y}}] = 0_{E_1,\sigma}^! [\mathcal{O}_{C_1}] \in K_0(X(\sigma)), \end{aligned}$$

using the cosection localized Gysin maps in intersection theory and K -theory respectively, where $[\mathcal{O}_{C_1}]$ can naturally be considered as an element of $K_0(E_1(\sigma))$.

To complete the proof, we need the cosection localized analogue of (5.12):

$$\tau_{X(\sigma)}([\mathcal{O}_{X,\text{loc}}^{\text{vir}}]) = \text{td}(-E_1) \cdot 0_{E_1,\sigma}^! (\tau_{E_1(\sigma)}([\mathcal{O}_{C_1}])).$$

But this is the special case $[F] = [\mathcal{O}_{C_1}]$ of formula (5.21) in the proof of [KL17, Theorem 5.8]. \square

An immediate corollary is the following cosection localized Riemann-Roch formula.

Theorem 6.2. *When the restriction $f' = f|_{X(\sigma)}: X(\sigma) \rightarrow Y$ is proper, we have for any $V \in K^0(X(\sigma))$*

$$\text{ch}(f'_*(V \otimes [\mathcal{O}_{X,\text{loc}}^{\text{vir}}])) \cdot \text{td}(T_Y) \cap [Y] = f'_*(\text{ch}(V) \cdot \text{td}^{\text{vir}}(X/Y) \cap [X]_{\text{loc}}^{\text{vir}}).$$

Suppose now that the group G is general. Our treatment of the equivariant Riemann-Roch theorem in Subsection 5.2 reduces the statement to the non-equivariant case. The following proposition allows us to follow the steps of Subsection 5.2 verbatim in the presence of a cosection.

Proposition 6.3. *Let (V, U) be an l -dimensional good pair for the integer $n - i$ and ϕ_U be the induced almost perfect obstruction theory for the morphism $f_U: X \times^G U \rightarrow Y \times^G U$ (cf. Proposition 5.11). Then, a G -invariant cosection σ for ϕ induces a canonically defined cosection $\sigma_U = s_U(\sigma)$ for ϕ_U with vanishing locus $(X \times^G U)(\sigma_U) = X(\sigma) \times^G U$.*

Proof. The statement follows immediately, since the obstruction sheaf of ϕ_U is $s_U(\mathcal{O}_{b_X})$ and we also have $s_U(\mathcal{O}_X) = \mathcal{O}_{X \times^G U}$. \square

Using Theorem 6.1, the proof of Theorem 5.14 goes through following the steps line by line to yield the following cosection localized virtual Riemann-Roch theorem.

Theorem 6.4. *Suppose that X admits a G -invariant closed embedding into a smooth algebraic space with a G -action and let $E^\vee = [E_0 \rightarrow E_1]$ be a G -equivariant global presentation of the virtual tangent bundle of f . Then*

$$\tau_{X(\sigma)}^G([\mathcal{O}_{X,\text{loc}}^{\text{vir}}]) = \frac{\text{td}^G(E_0)}{\text{td}^G(E_1)} \cdot \frac{\text{td}^G(f^*T_Y)}{\text{td}^G(\mathfrak{g})} \cap [X]_{\text{loc}}^{\text{vir}} \in A_*^G(X(\sigma)),$$

where $T_{X/Y}^{\text{vir}} = [E_0] - [E_1] + [f^*T_Y] - [\mathfrak{g}] \in K_G^0(X)$ and \mathfrak{g} is the adjoint representation of G .

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