STABILIZER REDUCTION FOR DERIVED STACKS AND APPLICATIONS TO SHEAF-THEORETIC INVARIANTS

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Abstract. We construct a canonical stabilizer reduction $\widetilde{X}$ for any derived 1-algebraic stack X over $\mathbb{C}$ as a sequence of derived Kirwan blow-ups, under mild natural conditions that include the existence of a good moduli space for the classical truncation X_{cl}. Our construction naturally generalizes Kirwan's classical partial desingularization algorithm to the context of derived algebraic geometry.

We prove that $\widetilde{X}$ is a natural derived enhancement of the intrinsic stabilizer reduction constructed by Kiem, Li and the third author. Moreover, if X is (-1)-shifted symplectic, we show that the semi-perfect and almost perfect obstruction theory and their induced virtual fundamental cycle and virtual structure sheaf of $\widetilde{X}_{\mathrm{cl}}$, constructed by the same authors, are naturally induced by $\widetilde{X}$ and its derived tangent complex. As a corollary, we give a fully derived perspective on generalized Donaldson–Thomas invariants of Calabi–Yau threefolds and define new generalized Vafa–Witten invariants for surfaces via Kirwan blow-ups.

Contents

1. Introduction

1.1. **Some brief history.** It has long been understood that in order to effectively study geometric objects and their families, one needs to enlarge the category of algebraic varieties and consider the more general notion of algebraic stacks. While the theory of algebraic stacks now enjoys a vast literature and features prominently within algebraic geometry and beyond,

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the presence of automorphisms of objects, also called stabilizers, which is typically a desirable feature, can sometimes be problematic regarding the use of standard techniques that apply to algebraic varieties, e.g., (co)homological methods like integration.

A very useful way to associate an algebraic space to an Artin stack is to consider its moduli space, whenever this makes sense. However, depending on the scope of one's considerations, this might not be adequate, given that it is not necessarily a well-behaved operation. For example, a smooth stack need not have a smooth moduli space.

Stabilizer reduction refers to the process of resolving the stackiness of an algebraic stack to produce a canonical Deligne–Mumford stack, whose stabilizers are all finite. In classical algebraic geometry, Kirwan [Kir85] was the first to carry this out for smooth quotient stacks $X = [Q^{\mathrm{ss}}/G]$ obtained by Geometric Invariant Theory [MFK94], namely by an action of a reductive group G on a smooth projective variety Q together with a linearization.

More precisely (assuming that $Q^s \neq \emptyset$), she produced a sequence of GIT quotient stacks

$$X_0 = X = [Q^{\mathrm{ss}}/G],\ X_1 = [Q_1^{\mathrm{ss}}/G],\ \dots\ ,\ \widetilde{X} \coloneqq X_n = [\widetilde{Q}^{\mathrm{ss}}/G]$$

by iteratively blowing up X_i along the (smooth) locus $X_i^{\max} \subseteq X_i$ of points with maximal dimensional stabilizer and then deleting unstable points by a careful study of stability on the blow-up. The final result $\widetilde{X} = [\widetilde{Q}^{\mathrm{ss}}/G]$ is a GIT quotient Deligne–Mumford stack satisfying $\widetilde{Q}^{\mathrm{ss}} = \widetilde{Q}^s$. At the level of GIT quotients, each morphism $X_{i+1} \to X_i$ fits into a natural commutative square

$$\begin{tikzcd} X_{i+1} = [Q_{i+1}^{\mathrm{ss}}/G] \arrow[r] \arrow[d] & X_i = [Q_i^{\mathrm{ss}}/G] \arrow[d] \\ Y_{i+1} = Q_{i+1}/\!\!/G \arrow[r] & Y_i = Q_i/\!\!/G \end{tikzcd} \tag{1.1}$$

where the lower arrow is a birational blow-up map. Thus, since $\widetilde{Y}$ only has finite quotient singularities and admits a projective, birational map $\widetilde{Y} \to Y$, it is a partial desingularization of Y. Kirwan's construction has had a wealth of applications, see for example [Kir86, KL04, Kie07].

Subsequently, Kirwan's construction has been independently generalized by Edidin–Rydh [ER21], and Kiem–Li and the third author [KLS17, Sav20], to apply to (possibly singular) Artin stacks X with a good moduli space $q\colon X \to Y$ in the sense of Alper [Alp13]. Such stacks are a natural generalization of GIT quotient stacks $X = [Q^{\mathrm{ss}}/G]$ with good moduli space morphism $q\colon [Q^{\mathrm{ss}}/G] \to Q/\!\!/G$. In fact, by deep results of Alper–Hall–Rydh [AHR19, AHR20], the GIT picture is the étale local model for this general class of stacks.

The two constructions, while related, differ in the notion of blow-up being applied at each iterated step: Edidin–Rydh introduce a saturated blow-up by first blowing up X along $X^{\max}$ and then carefully deleting a closed substack of unstable points, whereas Kiem–Li and the third author use a so-called Kirwan blow-up by first taking an intrinsic blow-up of X along $X^{\max}$ and then deleting unstable points. The intrinsic blow-up was constructed using local embeddings into smooth quotient stacks and preserved Kuranishi models. It was thus expected to be the classical shadow of a derived blow-up.

1.2. **Derived stabilizer reduction: statement of results.** In this paper, we generalize Kirwan's algorithm to the setting of derived algebraic stacks X whose classical truncation X_{cl} admits a good moduli space $q\colon X_{\mathrm{cl}} \to Y$. By the hidden smoothness principle, derived stacks should behave in analogy with smooth classical stacks, and hence it is natural to expect that a derived stabilizer reduction $\widetilde{X}$ should be the result of repeatedly blowing up X along $X^{\max}$ and then deleting unstable points and this is exactly what we do.

Our first result proves the existence of $X^{\max}$ in the derived setting, under mild finiteness assumptions which we leave implicit here and in the rest of the introduction. Uniqueness refers to uniqueness up to a contractible space of homotopies.

Theorem (Theorem 5.13)**.** *Let X be a derived Artin stack whose classical truncation admits a good moduli space $X_{\mathrm{cl}} \to Y$. Let d be the maximal dimension of stabilizers of points of X.*

Then there exists a unique, closed immersion $X^{\max} \to X$ of derived algebraic stacks such that for any étale morphism $[U/G] \to X$, with U an affine derived scheme and G a reductive group of dimension d, whose classical truncation fits in a Cartesian diagram

$$\begin{tikzcd} {[U_{\mathrm{cl}}/G]} \arrow[r] \arrow[d] & X_{\mathrm{cl}} \arrow[d] \\ U_{\mathrm{cl}}/\!\!/G \arrow[r] & Y, \end{tikzcd} \tag{1.2}$$

there exists a Cartesian diagram of derived stacks

$$\begin{tikzcd} {[U^{G^0}/G]} \arrow[r] \arrow[d] & {[U/G]} \arrow[d] \\ X^{\max} \arrow[r] & X, \end{tikzcd}$$

where $G^0 \subseteq G$ is the identity component of G. The classical truncation of $X^{\max}$ is naturally isomorphic to $(X^{\max})_{\mathrm{cl}}$.

For a classical stack, a precise treatment of $X^{\max}$ was given in the appendix of [ER21]. Our definition gives a derived enhancement, directly inspired by the classical construction. To prove Theorem 5.13 and to compare the derived blow-up of X along the locus $X^{\max}$ to the intrinsic blow-up, we use properties of derived fixed loci, which are defined via Weil restrictions. A substantial part of the paper is devoted to a thorough treatment of elements of derived equivariant geometry for this reason.

With $X^{\max}$ at hand, the recent theory of derived blow-ups in arbitrary derived centers—generalized by the first author in [Hek21] from the quasi-smooth case treated in [KR18], and further developed in [HKR22]—together with its natural extension to (equivariant) derived stacks, allows us to define the derived intrinsic blow-up $X^{\mathrm{intr}} \coloneqq \mathrm{Bl}_{X^{\max}} X$ of X in the center $X^{\max}$.

Our second result shows that its classical truncation is the intrinsic blow-up $(X_{\mathrm{cl}})^{\mathrm{intr}}$ of X_{cl} defined by Kiem–Li and the third author. We may then delete unstable points as in [KLS17] to obtain an open substack $\widehat{X} \subseteq X^{\mathrm{intr}}$, which we define to be the derived Kirwan blow-up of X.

We summarize these results in the following theorem.

Theorem (Theorem 6.3, Proposition 7.2, Theorem 7.4)**.** *There exists a canonical derived Artin stack $\widehat{X}$, called the derived Kirwan blowup of X, together with a morphism $\pi\colon \widehat{X} \to X$, such that:*

(i) *Its classical truncation $\widehat{X}_{\mathrm{cl}}$ admits a good moduli space morphism $\widehat{q}\colon \widehat{X}_{\mathrm{cl}} \to \widehat{Y}$.*
(ii) *The maximum stabilizer dimension of points in $\widehat{X}$ is strictly smaller than that of X.*
(iii) *For any affine, étale, stabilizer-preserving morphism $[U/G] \to X$ whose classical truncation fits in a Cartesian diagram* (1.2)*, the base change $\widehat{X} \times_X [U/G]$ is naturally isomorphic to the derived Kirwan blowup of $[U/G]$.*
(iv) *$\pi|_{\pi^{-1}(X^{\mathrm{s}})}$ is an isomorphism over the open locus X^{s} of stable points.*

$\widehat{X}$ is the semistable locus $(X^{\mathrm{intr}})^{\mathrm{ss}} \subseteq X^{\mathrm{intr}}$, an open substack of the Artin stack $X^{\mathrm{intr}} = \mathrm{Bl}_{X^{\max}} X$, called the derived intrinsic blowup of X.

The classical truncations of X^{intr} and $\widehat{X}$ are the classical intrinsic and Kirwan blow-ups of the classical truncation X_{cl} respectively.

The main ingredient in the proof is an explicit, non-trivial computation of the classical truncation of the equivariant blow-up of $\operatorname{Spec} A$ along $(\operatorname{Spec} A)^{G^0}$, using several properties of derived blow-ups and fixed loci. Since $(\mathrm{Bl}_{X^{\max}} X)_{\mathrm{cl}}$ is in general not the blow-up of X_{cl} along $X_{\mathrm{cl}}^{\max}$, this theorem explains the difference between the two constructions [ER21] and [KLS17]. When X is smooth, these do coincide and both blow-ups are the same.

As an immediate consequence of this theorem we get the desired derived stabilizer reduction $\widetilde{X} \to X$ of X by repeatedly taking derived Kirwan blow-ups until the maximal stabilizer dimension of points drops to zero. $\widetilde{X}$ is a derived Deligne–Mumford stack, which by construction is a derived enhancement of the intrinsic stabilizer reduction $\widetilde{X}_{\mathrm{cl}}$ of the classical truncation X_{cl} constructed in [Sav20].

1.3. (-1)**-shifted symplectic geometry and sheaf-theoretic invariants.** The main motivation for the introduction of the intrinsic stabilizer reduction $\widetilde{X}_{\mathrm{cl}}$ in [KLS17, Sav20] was the construction of intersection-theoretic generalized Donaldson–Thomas invariants.

Namely, it was shown by local computation that if X is a (-1)-shifted symplectic derived Artin stack, then the intrinsic stabilizer reduction $\widetilde{X}_{\mathrm{cl}}$ admits a canonically induced semi-perfect obstruction theory (cf. [CL11]) of virtual dimension zero, and later in [KS22] that in fact this can be promoted to an almost perfect obstruction theory. In particular, there are an associated virtual fundamental cycle and virtual structure sheaf

$$[\widetilde{X}_{\mathrm{cl}}]^{\mathrm{vir}} \in A_0(\widetilde{X}_{\mathrm{cl}}), \quad [\mathcal{O}^{\mathrm{vir}}_{\widetilde{X}_{\mathrm{cl}}}] \in K_0(\widetilde{X}_{\mathrm{cl}}), \tag{1.3}$$

and one can define

$$\mathrm{DTK}(X) = \int_{[\widetilde{X}_{\mathrm{cl}}]^{\mathrm{vir}}} 1 \in \mathbb{Q}$$

as the numerical generalized Donaldson–Thomas invariant via Kirwan blow-ups (DTK invariant) associated to X, along with K-theoretic invariants. Since derived moduli stacks of semistable sheaves and perfect complexes on Calabi–Yau threefolds are (-1)-shifted symplectic by [PTVV13], this gives rise to DTK invariants that act as virtual counts of these objects (after rigidifying scaling automorphisms).

By suitable local computation, we further show that our derived stabilizer reduction procedure gives a derived enhancement of $\widetilde{X}_{\mathrm{cl}}$ which completely recovers the above obstruction theory.

Theorem (Theorem 8.14). *Let X be a (-1)-shifted symplectic derived Artin stack and $\widetilde{X}$ its derived stabilizer reduction. Then:*

(i) *The $[0,1]$-truncation $E_\bullet = \tau^{[0,1]}\mathbb{T}_{\widetilde{X}}|_{\widetilde{X}_{\mathrm{cl}}}$ (with cohomological indexing notation) is a perfect complex.*

(ii) *Its dual $E^\bullet = (E_\bullet)^\vee$ together with the derived structure of $\widetilde{X}$ naturally recover the data of the semi-perfect obstruction theory of* [KLS17, Sav20] *and almost perfect obstruction theory of* [KS22] *and hence the virtual fundamental cycle $[\widetilde{X}_{\mathrm{cl}}]^{\mathrm{vir}}$ and virtual structure sheaf $[\mathcal{O}^{\mathrm{vir}}_{\widetilde{X}_{\mathrm{cl}}}]$ in* (1.3).

This gives a fully derived perspective on generalized Donaldson–Thomas invariants via Kirwan blow-ups. As a corollary, we may combine the DTK formalism with virtual torus localization (cf. [GP99]) in order to define new, generalized Vafa–Witten invariants enumerating Gieseker semistable Higgs pairs $(E, \phi\colon E \to E \otimes K_S)$ on S with fixed positive rank and Chern classes, fixed determinant $\det E$ and zero trace $\mathrm{tr}\ \phi = 0$ (see Definition 9.3).

1.4. **Future directions.** Given Kirwan's original algorithm and its generalization in [ER21], it is natural to wonder whether there is a theory of derived good moduli spaces for derived algebraic stacks, such that, in particular, our derived stabilizer reduction procedure gives a sequence of diagrams

$$\begin{tikzcd}
X_{i+1} = \widehat{X}_i \arrow[r] \arrow[d] & X_i \arrow[d] \\
Y_{i+1} \arrow[r] & Y_i,
\end{tikzcd}$$

where the vertical maps are derived good moduli space morphisms, and which are thus derived enhancements of the diagrams (1.2). In fact, using the derived étale slice theorem and local-to-global methods, one can indeed construct derived good moduli spaces whose existence is preserved by the operation of derived Kirwan blow-up. The details will appear elsewhere to keep the length of the paper under control.

Another interesting direction of inquiry would be to investigate the properties and degeneracy of the pullback of an n-shifted symplectic form to the derived stabilizer reduction and possibly develop a theory of n-shifted quasi-symplectic derived stacks, of which derived stabilizer reductions are a particular case.

Finally, natural extensions of this work would be to generalize the stabilizer reduction algorithm to derived stacks over a base of positive or even mixed characteristic. Since our main arguments ultimately depend on the classical situation and on the formal properties of Weil restrictions, we in fact expect a generalized stabilizer reduction algorithm to exist in any nonconnective algebraic geometry. Such geometries are extensions of derived algebraic geometry—of which derived analytic geometry in the sense of [BK17], [BM22], [BKK] is an example—and are proposed by Ben-Bassat and the first author in [BH22], together with a generalization of derived blow-ups to this setting.

We intend to come back to these questions in the future.

1.5. **Layout of the paper.** In §2 we review the requisite background and language of derived algebraic geometry that we use. In §3, we recall derived blow-ups and discuss and prove statements that are necessary in subsequent parts of the paper. §4 is devoted to a careful study of derived equivariant geometry with a focus on fixed loci of derived group actions. §5 gives the construction of the locus $X^{\max}$ of points of maximal stabilizers of a derived stack X. In §6 we establish the relationship between intrinsic blow-ups and equivariant derived blow-ups, and in §7 we construct the derived stabilizer reduction of a derived Artin stack. §8 deals with the case of (-1)-shifted symplectic stacks and, finally, §9 discusses applications to generalized Donaldson–Thomas and Vafa–Witten invariants.

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1.7. **Notation and conventions.** Throughout, we work over $\mathbb{C}$. All rings and algebras are assumed to be commutative. We will use both simplicial/derived rings and commutative differential graded algebras interchangeably, assuming their usual equivalence under the Dold–Kan correspondence. The particular model we are working with will be clear from context.

All derived stacks and all good moduli space morphisms are assumed to have affine diagonal. By a derived algebraic stack we mean a derived 1-algebraic stack.

The classical truncation of X is denoted by X_{cl} and the topological space of points of X by $|X|$. If $x \in |X|$, then G_x denotes the automorphism group/stabilizer of $x \in X_{\mathrm{cl}}$. In practice, we will mostly work with stabilizers of closed points $x \in |X|$. These will be reductive for most stacks of interest in this paper, namely stacks whose classical truncation admits a good moduli space.

For a morphism $\rho\colon X \to Y$ and a sheaf or complex $\mathcal{E}$ on Y, we often use $\mathcal{E}|_X$ to denote $\rho^*\mathcal{E}$, suppressing the pullback from the notation. We also use the phrases "closed embedding" and "closed immersion" interchangeably, likewise for "Artin stack" and "algebraic stack".

G, H denote complex reductive groups throughout. Since we are working over $\mathbb{C}$, being reductive is equivalent to being linearly reductive and hence all finite-dimensional and rational G-representations are completely reducible in this paper (see, for example, [Bor66, Theorem 5.2]). Usually, H will be a subgroup of G. G^0 is the identity component of G. T is the torus $\mathbb{C}^\times$.

If X is a derived stack with G-action, X^G is used to denote the derived fixed point locus of G in X, see §4.4. For X algebraic, $X^{\mathrm{intr}}, \widehat{X}$ are used to denote the derived intrinsic and Kirwan blow-ups of X respectively, see §7.2. $\widetilde{X}$ denotes the derived stabilizer reduction of a stack X, see §7.3.

Quasi-coherent modules and algebras are written as $\mathcal{M}, \mathcal{N}, \ldots$ and $\mathcal{A}, \mathcal{B}, \ldots$ to distinguish them from their affine counterparts $M, N, \ldots$ and $A, B, \ldots$.

We will use homological indexing notation, unless otherwise stated.

We will mostly ignore set-theoretic issues to avoid distractions. These can be resolved by any of the standard methods, such as assuming the existence of Grothendieck universes.

2. Recollections on derived algebraic geometry

We review the language of derived algebraic geometry which we will be using. Much of this material can be found in the standard literature on the topic, like [SAG], [HAGII], [GR17]. For convenience, we collect some constructions and results here, but refer the reader to these sources for more details.

2.1. Background on ∞-categories.

Recall that an ∞-category is a generalization of a 1-category that allows us to talk about n-morphisms for all $n \in \mathbb{N}_{\geq 1}$, where all morphisms above $n = 1$ are invertible (in other words, these are $(\infty, 1)$-categories). We thus have, between any two objects x, y in a given ∞-category C, a mapping space $\mathsf{C}(x, y)$ which has the set of equivalence classes of 1-morphisms $x \to y$ as connected components. Here, a space is considered as a homotopy type, which can be modelled for example on simplicial sets.

We use similar shorthands for ∞-categories as is customary for 1-categories, by expressing the existence of a homotopy as a property. In particular, when we say that a diagram is commutative, we mean that there is a homotopy that makes the diagram commutative. Likewise, by uniqueness we will mean uniqueness up to a contractible space of choices.

Write Cat for the ∞-category of ∞-categories, and Spc for the ∞-category of spaces. Although we will explicitly distinguish ∞-categories from 1-categories throughout, we do adopt the common language of ∞-categories, where limits, colimits, Kan extensions, etc., are always taken in their ∞-categorical meaning. Recall, if one chooses a point-set model, then these become homotopy limits, colimits, Kan extensions, etc..

Definition 2.1. Let C be an ∞-category. Define the *cocompletion of* C *under sifted colimits*, written $\mathsf{P}_\Sigma(\mathsf{C})$, as the full subcategory of the ∞-category $\mathsf{P}(\mathsf{C})$ of presheaves on C spanned by those presheaves that send finite coproducts in C to products in Spc.

Remark 2.2. The ∞-category $\mathsf{P}_\Sigma(\mathsf{C})$ is the universal way of adding sifted colimits to C. It is also called the *nonabelian derived category* of C. See [HTT, §5.5.8] for details.

2.2. **Derived rings and modules.** Let Mod be the symmetric monoidal ∞-category of modules over $\mathbb{C}$ in the stably homotopic sense, i.e., of spectra endowed with a $\mathbb{C}$-action. Endow Mod with the standard t-structure (with homological indexing convention). Recall that Mod is also the ∞-category associated to the unbounded derived category of $\mathbb{C}$-vector spaces.

Write Mod^0 for the full subcategory of Mod spanned by the discrete, finitely generated $\mathbb{C}$-modules. Then Mod^0 generates $\mathsf{Mod}_{\geq 0}$ under sifted colimits.

Definition 2.3. Let Alg be the ∞-category of connective $\mathbb{E}_\infty$-algebras in Mod.

Let Poly be the category of finitely generated polynomial rings over $\mathbb{C}$. Since we are working over a ring of characteristic 0, Alg is equivalent to the ∞-category $\mathsf{P}_\Sigma(\mathsf{Poly})$, hence also to the ∞-category associated to the model category of simplicial $\mathbb{C}$-algebras, endowed with the projective model structure. It is also equivalent to the ∞-category of cdgas over $\mathbb{C}$ concentrated in homological degree ≥ 0.

An object R of Alg will be called an *algebra*. We can associate to R a 1-categorial model R', either incarnated as a simplicial ring or as a cdga. Of course, any R'' which is weakly equivalent to R' will then also be a 1-categorical model of R. To keep the indexing convention consistent, we will write our cdgas as chain complexes in degree ≥ 0.

The forgetful functor $\mathsf{Alg} \to \mathsf{Mod}_{\geq 0}$ has a left adjoint, written LSym. For $R \in \mathsf{Alg}$, we write Alg_R for the slice-∞-category $\mathsf{Alg}_{R/}$. Objects in Alg_R are called *R-algebras.* Likewise, we write Mod_R for the ∞-category of R-modules in Mod. The forgetful functor $\mathsf{Alg}_R \to (\mathsf{Mod}_R)_{\geq 0}$ again has a left adjoint, written LSym_R.

Let Mod_R^0 be the full subcategory of Mod_R spanned by finitely generated free R-modules, and let $\mathsf{Poly}_R \subset \mathsf{Alg}_R$ be the essential image of LSym_R on Mod_R^0. Then Mod_R^0 again generates $(\mathsf{Mod}_R)_{\geq 0}$ under sifted colimits, and the canonical map $\mathsf{P}_\Sigma(\mathsf{Poly}_R) \to \mathsf{Alg}_R$ is an equivalence. Moreover, Alg_R is the ∞-category of connective $\mathbb{E}_\infty$-algebras in Mod_R.

The functor $\pi_0 \colon \mathsf{Spc} \to \mathsf{Set}$ induces adjunctions

$$\pi_0 : \mathsf{Mod}_{\geq 0} \leftrightarrows \mathsf{Mod}^\heartsuit : i \qquad\qquad \pi_0 : \mathsf{Alg} \leftrightarrows \mathsf{Alg}^\heartsuit : i$$

where $\mathsf{Mod}^\heartsuit$ is the heart of Mod with respect to the given t-structure, i.e., the 1-category of $\mathbb{C}$-vector spaces, and $\mathsf{Alg}^\heartsuit$ is the 1-category of discrete $\mathbb{C}$-algebras.

For $R \in \mathsf{Alg}$ and $M \in \mathsf{Mod}_R$, the homotopy groups $\pi_n(M)$ are canonically $\pi_0(R)$-modules, for all $n \in \mathbb{Z}$. In particular, $\pi_n(R)$ is a $\pi_0(R)$-module for all $n \geq 0$.

2.3. **Cell attachments.** Let $R \in \mathsf{Alg}$. For $n \geq 0$, we adjoin a free variable in homological degree n to R by writing $R[u] = R[S^n]$, which has the universal property that the space of R-algebra maps $R[u] \to B$ is equivalent to the space of maps $(S^n, *) \to (B, 0)$ of pointed spaces, i.e., to $\mathsf{Mod}_R(R[n], B)$.

Let a sequence $\underline{\sigma} = (\sigma_1, \dots, \sigma_k)$ of elements $\sigma_i \in \pi_{n_i}(R)$ be given. Then we define the *finite quotient* of R by σ, written $R/(\underline{\sigma}) = R/(\sigma_1, \dots, \sigma_k)$, via the pushout

$$\begin{tikzcd} R[u_1, \dots, u_k] \arrow[r, "z"] \arrow[d, "s"] & R \arrow[d] \\ R \arrow[r] & R/(\sigma_1, \dots, \sigma_k) \end{tikzcd}$$

where the u_i are free in homological degree n_i, the map s is induced by lifts $S^{n_i} \to R$ of σ_i, and z sends each u_i to zero. When $k = 1$, we also say that $R/(\underline{\sigma})$ is obtained from R by *attaching an* $(n+1)$*-cell.* A *zero cell attachment* is the map $R \to R[t]$, where t is in homological degree 0. Note that the input of attaching several $(n+1)$-cells is equivalent to a morphism $\sigma \colon M[n] \to R$ of R-modules, where M is free of finite rank. Giving this datum, we write the quotient also as $R/(\sigma)$.

A finite quotient of the form $R \to R/(f_1, \dots, f_k)$, where each f_i is in homological degree 0, is called *quasi-smooth.*

2.4. **Standard forms.** For the purpose of performing computations, it will be useful to work with certain 1-categorical models.

Recall that a morphism $A \to B$ of algebras (and the corresponding map on spectra) is *locally of finite presentation* if B is a compact object in Alg_A, i.e., the Spc-valued functor $\mathsf{Alg}_A(B, -)$ preserves filtered colimits. It is *finitely presented* if B can be obtained from A by a finite number of cell attachments. Then the following are equivalent: (1) B is locally of finite presentation over A; (2) the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is, Zariski-locally on $\operatorname{Spec} B$, finitely presented; and (3) B is a retract of a finitely presented A-algebra.

Definition 2.4. Let A be an algebra. A cdga model R for A is *in standard form* if R_0 is smooth with Ω_{R_0} free, and the underlying graded-commutative ring of R is freely generated over R_0 by a finite number of generators.

Remark 2.5. Observe, if a cdga R is in standard form, then it can be obtained from R_0 by a finite number of cell-attachments, say $R(0) \to R(1) \to \cdots \to R(n)$, in such a way that when we terminate the process at $k \leq n$, then $R(k)$ is also in standard form. In fact, the underlying graded-commutative ring of $R(k)$ is the subring of the underlying graded-commutative ring of R generated by elements of homogeneous degree $\leq k$, and $R(k+1) = R(k)/(\sigma)$ where $\sigma \colon M_{k+1}[k] \to R(k)$ and M_{k+1} is a free module of finite rank over $R(k)$. Moreover $R_{k+1} = R(k)_{k+1} \oplus (M_{k+1})_0$. We say that R is *generated* by the M_{k+1}.

If R is a model of standard form for A, then the cotangent complex of $\operatorname{Spec} A$ is given by the Kähler differentials Ω_R, see [BBJ19, §2.3].

Definition 2.6. Let $x \in \operatorname{Spec} A$ with $A \in \mathsf{Alg}$ be given. Then a model R in standard form for A is *minimal at x* if the differentials of $\Omega_R|_x$ are all zero.

By the following lemma, locally finitely presented derived schemes can Zariski-locally be described by models in standard form.

Lemma 2.7. *Let $A \in \mathsf{Alg}$ be locally of finite presentation and $x \in \operatorname{Spec} A$. Then, up to Zariski localizing around x, we can find a model R in standard* *form for A such that R_0 is a finitely generated polynomial ring, or such that it is minimal at x.*

Proof. This follows for example from [BBJ19, Theorem 4.1]. □

Remark 2.8. In general, the two conditions cannot be met simultaneously, see [BBJ19, Remark 3.2].

2.5. **Graded derived rings.** Derived blow-ups in arbitrary centers are defined along similar lines as the classical construction—namely, through a derived version of the Rees algebra, of which one then takes the projective spectrum. To explain this, let us first review graded algebras in the derived setting as defined in [Hek21].

Let $\mathbb{M}$ be a commutative monoid, considered as a discrete symmetric monoidal category in the obvious way. Let $\mathsf{Mod}^{\mathbb{M}}$ be the functor ∞-category $\mathrm{Fun}(\mathbb{M}, \mathsf{Mod})$, endowed with symmetric monoidal structure via Day convolution and with t-structure inherited from Mod. An object M of $\mathsf{Mod}^{\mathbb{M}}$ is called an $\mathbb{M}$-*graded module*, written as $\bigoplus_{d \in \mathbb{M}} M_d$, where M_d is M evaluated at d.

For $d \in \mathbb{M}$ and $N \in \mathsf{Mod}$, we write $N(d)$ for the $\mathbb{M}$-graded module which is N concentrated in homogeneous degree d. Let then $(\mathsf{Mod}^{\mathbb{M}})^0$ be the full subcategory of $\mathsf{Mod}^{\mathbb{M}}$ generated by finite coproducts of modules of the form $N(d)$, where $N \in \mathsf{Mod}^0$ and $d \in \mathbb{M}$. Then $(\mathsf{Mod}^{\mathbb{M}})^0$ generates $(\mathsf{Mod}^{\mathbb{M}})_{\geq 0}$ under sifted colimits.

Define the ∞-category of $\mathbb{M}$-*graded algebras*, written $\mathsf{Alg}^{\mathbb{M}}$, as the ∞-category of connective $\mathbb{E}_\infty$-algebras in $\mathsf{Mod}^{\mathbb{M}}$. As in the ungraded case, the forgetful functor $\mathsf{Alg}^{\mathbb{M}} \to (\mathsf{Mod}^{\mathbb{M}})_{\geq 0}$ has a left adjoint, written $\mathrm{LSym}^{\mathbb{M}}$. Let $\mathsf{Poly}^{\mathbb{M}}$ be the essential image of $\mathrm{LSym}^{\mathbb{M}}$ evaluated on $(\mathsf{Mod}^{\mathbb{M}})^0$. Then the canonical map $\mathsf{P}_\Sigma(\mathsf{Poly}^{\mathbb{M}}) \to \mathsf{Alg}^{\mathbb{M}}$ is an equivalence.

A key result from [Hek21] is that, as expected, the ∞-category $\mathsf{Alg}^{\mathbb{Z}}$ is (contravariantly) equivalent to the ∞-category of affine derived schemes with $\mathbb{G}_m$-action. The general theory of ∞-actions is reviewed in §3.1, and we give an in-depth analysis of G actions on derived stacks for reductive algebraic groups G in §4.

For $B \in \mathsf{Alg}^{\mathbb{M}}$, write $\mathsf{Mod}_B^{\mathbb{M}}$ for the ∞-category of $\mathbb{M}$-*graded B-modules*, which by definition are B-modules in $\mathsf{Mod}^{\mathbb{M}}$. As in the ungraded case, $\mathsf{Mod}_B^{\mathbb{M}}$ is stable and symmetric monoidal, and $(\mathsf{Mod}_B^{\mathbb{M}})_{\geq 0}$ is generated under sifted colimits by the ∞-category of finitely generated, free, $\mathbb{M}$-graded B-modules. Once again, $\mathsf{Alg}_B^{\mathbb{M}} := (\mathsf{Alg}^{\mathbb{M}})_{B/}$ is generated under sifted colimits by the finitely generated, $\mathbb{M}$-graded polynomial B-algebras, which are defined similarly as in the ungraded case.

We are primarily interested in $\mathbb{N}$-graded and $\mathbb{Z}$-graded rings. These exhibit familiar behavior: to a $\mathbb{Z}$-graded ring B we can associate an $\mathbb{N}$-graded ring $B_{\geq 0}$ by discarding the pieces in negative homogeneous degrees, and an $\mathbb{N}$-graded ring B' can be promoted to a $\mathbb{Z}$-graded ring by putting 0 in negative homogeneous degrees. We can also take the homogeneous degree-zero part B_0 of a $\mathbb{Z}$-graded ring, which gives us morphisms $B_0 \to B_{\geq 0} \to B$ of $\mathbb{Z}$-graded rings (where we think of $B_0, B_{\geq 0}$ as $\mathbb{Z}$-graded by the procedure just described, which we suppress from notation). If B is $\mathbb{N}$-graded, we also have a morphism $B \to B_0$ such that the composition $B_0 \to B \to B_0$ is homotopic to the identity.

Let $\mathbb{M}$ be either $\mathbb{N}$ or $\mathbb{Z}$, and let $B \in \mathsf{Alg}^{\mathbb{M}}$. As in the ungraded case, we can adjoin a free variable to B in homological degree $n \in \mathbb{N}$ and homogeneous degree $d \in \mathbb{M}$, which produces an $\mathbb{M}$-graded B-algebra $B[u]$ with the universal property that the space of $\mathbb{M}$-graded B-algebra maps $B[u] \to C$ is equivalent to the space of maps $(S^n, *) \to (C_d, 0)$, i.e., to $\mathsf{Mod}^{\mathbb{M}}(B[n]_d, C)$, where $B[n]_d$ is the homogeneous degree-d part of the module $B[n]$.

The $\mathbb{M}$-grading on B induces an $\mathbb{M}$-grading on each $\pi_n(B)$. We can thus define a *finite quotient* of B by a sequence $\underline{\sigma} = (\sigma_1, \dots, \sigma_k)$ of elements $\sigma_i \in \pi_{n_i}(B)_{d_i}$, with $n_i \in \mathbb{N}$ and $d_i \in \mathbb{M}$, using the graded free variables construction just described and performing cell attachments in the same way as in the ungraded case. The result is again written as $B/(\underline{\sigma})$.

2.6. **Derived stacks.** Let $\mathbf{P}$ be one of the following properties: étale, a (Zariski) open immersion, smooth. A morphism $A \to B$ of derived rings is called $\mathbf{P}$ if $\pi_0 A \to \pi_0 B$ is $\mathbf{P}$, and the morphism $(\pi_n A) \otimes_{\pi_0 A} (\pi_0 B) \to \pi_n B$ is an isomorphism, for any $n \geq 0$. Define the ∞-category of *affine derived schemes* as $\mathsf{Aff} \coloneqq \mathsf{Alg}^{\mathrm{op}}$, and write $\mathrm{Spec}(-)\colon \mathsf{Alg}^{\mathrm{op}} \to \mathsf{Aff}$ for the obvious functor. Declare morphisms in Aff to be $\mathbf{P}$ if the corresponding map in Alg is $\mathbf{P}$. For $U = \mathrm{Spec}\, A$ in Aff, the *underlying classical scheme* is $U_{\mathrm{cl}} \coloneqq \mathrm{Spec}\, \pi_0 A$.

We endow Aff with the *étale topology* by declaring a family $\{U_i \to U\}_i$ in Aff to be a cover if all $U_i \to U$ are étale, and the induced maps on classical schemes $(U_i)_{\mathrm{cl}} \to U_{\mathrm{cl}}$ are jointly surjective.

A *derived prestack* is a functor $X\colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Spc}$. A *derived stack* is a derived prestack which is a sheaf (in the ∞-categorical sense) for the étale topology. We write Stk for the ∞-category of derived stacks. For $X \in \mathsf{Stk}$, we write Stk_X for the ∞-category $\mathsf{Stk}_{/X}$ of derived stacks over X. The Yoneda embedding induces an embedding $\mathrm{Spec}(-)\colon \mathsf{Aff} \to \mathsf{Stk}$.

Recall that a space K is called *n-truncated* if $\pi_m(K, x)$ is trivial for all points $x \in K$ and all $m > n$. A stack X is called *n-truncated* if $X(T)$ is n-truncated for all classical affine schemes T. A morphism $X \to Y$ of derived stacks is *affine* if $X \times_Y (\mathrm{Spec}\, A)$ is affine, for all $\mathrm{Spec}\, A \to Y$. Then a derived stack X has *affine diagonal* if the canonical morphism $\Delta\colon X \to X \times X$ is affine. Observe that X has affine diagonal if and only if the intersection $U \times_X V$ is affine, for any pair of affine derived schemes U, V over X.

Assumption 2.9. From here on, assume that all derived stacks are 1-truncated and have affine diagonal, unless otherwise stated.

Let clStk be category of classical stacks. Composition with the inclusion $\mathsf{Alg}^{\heartsuit} \to \mathsf{Alg}$ gives a functor $(-)_{\mathrm{cl}}\colon \mathsf{Stk} \to \mathsf{clStk}$ which sends $X \in \mathsf{Stk}$ to the *underlying classical stack* X_{cl}. This functor has a fully faithful left adjoint ι, and both functors $\iota, (-)_{\mathrm{cl}}$ preserve affines. A morphism $X \to Y$ of derived stacks is *quasi-compact/a closed immersion* if the morphism $X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$ is quasi-compact/a closed immersion.

2.7. **Derived schemes.** In this and the next paragraph, let $\mathbf{P}$ be one of the properties: étale, smooth. Although we could add here any property which is étale-local in an approporiate sense, we restrict ourselves to what we need.

Let X be a derived stack. A family $\{U_i \to X\}_i$ of morphisms in Stk is a *cover* if the canonical map $\operatorname{colim} \check{C}(f) \to X$ is an equivalence, where $\check{C}(f)$ is the Čech nerve of $f\colon \coprod U_i \to X$, and the colimit is taken in Stk. A morphism $T \to X$ in Stk with affine source *is* $\mathbf{P}$ if $S \times_X T \to S$ is $\mathbf{P}$ for all affine derived scheme S over X. The latter is well-defined, since we assume that all derived stacks have affine diagonal. We say that $T \to X$ is an *open immersion* if it is an étale monomorphism.

A derived stack U is called a *derived scheme* if there is cover $\{U_i \to U\}$ of open immersions $U_i \to U$ by affine derived schemes U_i. Observe, the adjunction $\iota : \mathsf{clStk} \leftrightarrows \mathsf{Stk} : (-)_{\mathrm{cl}}$ restricts to an adjunction between classical schemes (which we assume to also have affine diagonal) and derived schemes. The ∞-category Sch of derived schemes is closed under fiber products in Stk.

A morphism $U \to X$ of derived stacks with U a derived scheme *is* $\mathbf{P}$ if $U \times_X V \to V$ is $\mathbf{P}$, for any derived scheme V over X.

2.8. **Derived algebraic spaces and stacks.** A *derived algebraic space* is a 0-truncated derived stack Y for which there is an étale cover $U \to Y$ by a derived scheme U. A morphism $X' \to X$ of derived stacks is *representable* if for all derived schemes U over X the fiber product $U \times_X X'$ is a derived algebraic space. Observe that any morphism $Y \to X$ from a derived algebraic space to a derived stack is representable. Again, this is because X has affine diagonal, and since for any affine morphism $X' \to V$ in Stk with V a derived scheme it holds that X' is a derived scheme as well.

A morphism $Y' \to Y$ of algebraic spaces *is* $\mathbf{P}$ if there is an étale cover $V \to Y'$ by a derived scheme V such that the composition $V \to Y$ is $\mathbf{P}$. A representable morphism $X' \to X$ of derived stacks *is* $\mathbf{P}$ if for any derived scheme U over X, the map $U \times_X X' \to U$ is $\mathbf{P}$.

A *derived algebraic stack* is a derived stack X such that there is a smooth cover $Y \to X$ with Y a derived algebraic space. Observe, the adjunction $\iota : \mathsf{clStk} \leftrightarrows \mathsf{Stk} : (-)_{\mathrm{cl}}$ restricts to an adjunction between classical algebraic stacks and derived algebraic stacks, and also between classical algebraic spaces and derived algebraic spaces.

Let $X' \to X$ be a morphism of algebraic stacks. Then $X' \to X$ is *smooth* if there is a smooth cover $Y' \to X'$ by a derived algebraic space Y' such that the composition $Y' \to X$ is smooth. Likewise, $X' \to X$ is *étale* if, smooth-locally on X, there is an étale cover $Y' \to X'$ by a derived algebraic space Y', such that the composition $Y' \to X$ is étale. Finally, $X' \to X$ is an *open immersion* if it is representable, étale and a monomophism.

2.9. **Quasi-coherent modules and algebras.** Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism of affine derived schemes. Then we have an adjunction $f^* : \mathsf{Mod}_A \leftrightarrows \mathsf{Mod}_B : f_*$, where f^* sends $M \in \mathsf{Mod}_A$ to $M \otimes_A B$. This gives us a functor $\mathsf{Aff}^{\mathrm{op}} \to \mathsf{Cat}$ that sends $\operatorname{Spec} R$ to Mod_R. Taking the right Kan extension yields the functor

$$\mathsf{Stk}^{\mathrm{op}} \to \mathsf{Cat}\colon X \mapsto \mathrm{QCoh}(X)$$

which sends X to the ∞-category of *quasi-coherent $\mathcal{O}_X$-modules.* Concretely, for $X \in \mathsf{Stk}$, we can write $\mathrm{QCoh}(X)$ as the limit of the diagram $\operatorname{Spec} R \mapsto \mathsf{Mod}_R$ over the comma ∞-category $\mathsf{Aff}^{\mathrm{op}}_{/X}$ consisting of derived affine schemes over X. This description tells us that we can think of a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ as a homotopy coherent diagram $\{\mathcal{F}_A\}_{\operatorname{Spec} A \to X}$, where each $\mathcal{F}_A$ is in Mod_A. We also write $\mathcal{F}(A) := \mathcal{F}(\operatorname{Spec} A) := \mathcal{F}_A$.

For $X \in \mathsf{Stk}$ algebraic, we can restrict the indexing category in the formula $\mathrm{QCoh}(X) \simeq \lim_{\operatorname{Spec} R \to X} \mathsf{Mod}_R$ to derived affine schemes which are smooth over X, and still get the same result. Then $\mathrm{QCoh}(X)$ is naturally a stable, presentable, symmetric monoidal ∞-category for which the tensor product commutes with colimits in each variable separately, and moreover $\mathrm{QCoh}(X)$ carries a t-structure such that $\mathcal{F} \in \mathrm{QCoh}(X)$ is connective if and only if $\mathcal{F}_A$ is, for all $\operatorname{Spec} A \to X$. For a morphism $f \colon X \to Y$ of derived algebraic stacks, by the adjoint functor theorem we have an adjunction $f^* : \mathrm{QCoh}(Y) \leftrightarrows \mathrm{QCoh}(X) : f_*$, where f^* is symmetric monoidal and f_* is right-lax symmetric monoidal.

Following the same procedure, we have a functor $\mathsf{Aff}^{\mathrm{op}} \to \mathsf{Cat}$ which sends $\operatorname{Spec} R$ to Alg_R. Right-Kan extending this yields a functor $\mathsf{Stk}^{\mathrm{op}} \to \mathsf{Cat}$, which we write as $X \mapsto \mathrm{QAlg}(X)$. Objects in $\mathrm{QAlg}(X)$ are called *quasi-coherent $\mathcal{O}_X$-algebras.* When restricted to derived algebraic stacks, this functor lands in the ∞-category of presentable, symmetric monoidal ∞-categories, and symmetric monoidal colimit preserving functors between them. Since we are in characteristic zero, we can also describe $\mathrm{QAlg}(X)$ as the ∞-category of $\mathbb{E}_\infty$-algebras in $\mathrm{QCoh}(X)_{\geq 0}$. We furthermore have a left adjoint to the forgetful functor $\mathrm{QAlg}(X) \to \mathrm{QCoh}(X)_{\geq 0}$, written LSym_X.

Example 2.10. For $X \in \mathsf{Stk}$, let $\mathcal{O}_X \in \mathrm{QAlg}(X)$ be such that $\mathcal{O}_X(A) = A$, for all $\operatorname{Spec} A \to X$, which is a unit in $\mathrm{QAlg}(X)$. We also write $\mathcal{O}_X$ for the underlying quasi-coherent module.

Let $X \in \mathsf{Stk}$ be algebraic. For $\mathcal{A} \in \mathrm{QAlg}(X)$, the (relative) *spectrum* of $\mathcal{A}$ is the derived algebraic stack $\operatorname{Spec} \mathcal{A}$ over X such that for $f \colon T \to X$, the space $(\operatorname{Spec} \mathcal{A})(T)$ is the space of quasi-coherent $\mathcal{O}_T$-algebra maps $f^*\mathcal{A} \to \mathcal{O}_T$.

Write Aff_X for the full subcategory of Stk_X spanned by derived stacks which are affine over X, i.e., for which the structure map is affine. For $\mathcal{A} \in \mathrm{QAlg}(X)$, it holds that $\operatorname{Spec} \mathcal{A}$ is affine over X. In fact, the functor $\mathcal{A} \mapsto \operatorname{Spec} \mathcal{A}$ yields an equivalence $\mathrm{QAlg}(X)^{\mathrm{op}} \simeq \mathsf{Aff}_X$.

Let $X \in \mathsf{Stk}$ be algebraic, and let $\mathcal{E} \in \mathrm{QCoh}(X)_{\geq(-1)}$ be given. Define $\mathbb{V}(\mathcal{E})$ as the derived stack over X with the universal property that for $f \colon T \to X$ it holds that $\mathbb{V}(\mathcal{E})$ is the space of $\mathcal{O}_T$-module maps $f^*\mathcal{E} \to \mathcal{O}_T$. If $\mathcal{E}$ is connective, then $\mathbb{V}(\mathcal{E}) \simeq \operatorname{Spec} \mathrm{LSym}_X(\mathcal{E})$.

2.10. **The cotangent complex and the normal bundle.** Let X be a derived algebraic stack. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is called *almost connective* if for all $\operatorname{Spec} A \to X$ there is some $n \in \mathbb{Z}$ such that $\mathcal{F}_A \in (\mathsf{Mod}_A)_{\geq n}$.

Let $T = \operatorname{Spec} A$ be an affine derived scheme, and M a connective A-module. Then there is a derived version of the square-zero extension of A by M, written $A \oplus M$, which is an A-algebra over A. We write $T[M]$ for $\operatorname{Spec}(A \oplus M)$.

Let $f \colon X \to Y$ be a morphism of derived algebraic stacks. The *cotangent complex* of f is the (-1)-connective, quasi-coherent $\mathcal{O}_X$-module $\mathcal{L}_{X/Y}$ with the following universal property. For any $f\colon T \to X$ with T affine, and any connective $\mathcal{O}_T$-module M, the space of maps $f^*\mathcal{L}_{X/Y} \to M$ of $\mathcal{O}_T$-modules is equivalent to the space of diagram fillers

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ T[M] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Let $X \xrightarrow{f} Y \to Z$ be a morphism of derived algebraic stacks. Then we have an exact sequence $f^*\mathcal{L}_{Y/Z} \to \mathcal{L}_{X/Z} \to \mathcal{L}_{X/Y}$ in $\mathrm{QCoh}(X)$. Moreover, for $X' \to Y'$ obtained by pulling back a derived algebraic stack Y' over Y along f, it holds that $h^*\mathcal{L}_{X/Y} \simeq \mathcal{L}_{X'/Y'}$, where h is the structure map $X' \to X$.

For $f\colon X \to Y$ a morphism of derived algebraic stacks, we let $\mathcal{N}_{X/Y}$ be the shifted cotangent complex $\mathcal{L}_{X/Y}[-1]$, and define the *normal bundle* of f as $\mathbb{N}_{X/Y} := \mathbb{V}(\mathcal{N}_{X/Y})$. Note that $\mathbb{V}(\mathcal{N}_{X/Y})$ will in general be 2-truncated, contrary to our standing assumption. This does not play a role in what follows.

3. Overview of derived blow-ups

We continue with developing the background for our main results, but specify to the recent theory on derived blow-ups and derived Rees algebras as developed in [KR18], [Hek21], [HKR22] and [BH22].

3.1. **∞-group actions.** Although the construction of the derived blow-up only uses G-actions on derived stacks for reductive algebraic groups G—which we analyse in-depth in §4—we will use actions by ∞-groups in Stk in §5 for the construction of $X^{\max}$. In this paragraph, we therefore review the parts of this theory that we will be using from [NSS15], [HTT]. We omit proofs since we only deviate slightly from the literature in terminology and notation.

Throughout, let H be an ∞-topos. Although we will only use the case $\mathsf{H} = \mathsf{Stk}$, presenting the theory in full generality highlights the formal character of the story.

Definition 3.1. Let $X\colon \Delta^{\mathrm{op}} \to \mathsf{H}$ be a simplicial diagram in H. Then X satisfies the *Segal condition* if, for all $n \geq 0$, the natural map

$$\varphi_n\colon X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

is an equivalence, where the morphisms φ_n come about from the spine inclusions $(0 \to 1 \to \cdots \to n) \to \Delta[n]$. If this is the case, then we say that *morphisms are invertible* in X if the map $X_2 \to X_1 \times_{X_0} X_1$ induced by the horn inclusion $\Lambda^2[2] \to \Delta[2]$ is an equivalence.

X is called an *∞-groupoid* (in H) if it satisfies the Segal condition such that morphisms are invertible. If moreover $X_0 \simeq *$, then X is called an *∞-group* (in H). Write $\mathrm{Grp}(\mathsf{H})$ for the full subcategory of $\mathsf{H}^{\Delta^{\mathrm{op}}}$ spanned by ∞-groups in H.

For G an ∞-group, we think of G_1 as the underlying object of G, and often conflate the two in our notation. We write $B(*, G)$ to specifically mean the full simplicial diagram $[n] \mapsto G_n$.

Example 3.2. An ∞-group(oid) in Set is a group(oid) in the classical sense. An ∞-group in Spc is a loop space, which one can think of as a topological group where the group structure is relaxed in a homotopy coherent way. If H is the category of derived stacks on an ∞-site C, then $\mathrm{Grp}(\mathsf{H})$ is equivalent to $\mathrm{Grp}(\mathsf{Spc})$-valued stacks on C. In particular, classical group stacks are ∞-groups in Stk.

For an ∞-category C, write $\mathrm{Arr}(\mathsf{C})$ for the arrow category $\mathrm{Fun}(\Delta[1], \mathsf{C})$, and let $\mathrm{Epi}(\mathsf{C})$ be the full subcategory of $\mathrm{Arr}(\mathsf{C})$ spanned by the effective epimorphisms. For an object T in C, write $\mathrm{Epi}_{T/}(\mathsf{C})$ for the full subcategory of $\mathrm{Epi}(\mathsf{C})$ spanned by morphisms with source T.

Proposition 3.3. *∞-groupoids in H are effective, meaning that the Čech nerve construction gives an adjoint equivalence of ∞-categories $\check{C}\colon \mathrm{Epi}(\mathsf{H}) \simeq \mathrm{Grpd}(\mathsf{H})$, with left adjoint induced by taking the colimit. Restricting to $\mathrm{Grp}(\mathsf{H})$ gives an adjoint equivalence*

$$B : \mathrm{Grp}(\mathsf{H}) \rightleftarrows \mathrm{Epi}_{*/}(\mathsf{H}) : \Omega$$

where $\Omega(* \to X)$ is the derived based loop stack $* \times_X *$.

Let G be an ∞-group in H and X an object in H. Then a *G-object* in H with underlying object X is an ∞-groupoid $B(X, G)$ over $B(*, G)$ of the form $[n] \mapsto G^n \times X$ such that $d_1\colon G \times X \to X$ is the projection. Informally, we say that X is endowed with a G-action $\sigma\colon G \times X \to X$ if there is a G-object $B(X, G)$ such that $d_0 \simeq \sigma$, and refer to $B(X, G)$ as the bar-construction of σ.

The ∞-category of G-objects is the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{H})_{B(*,G)}$ spanned by G-objects. Informally, if X, Y are endowed with a G-action, then a *G-equivariant morphism* $f\colon X \to Y$ is a morphism of the corresponding G-objects $f_\bullet\colon B(X, G) \to B(Y, G)$ such that $f_1 \simeq f$. For a G-object $B(X, G)$, we write the colimit of the diagram $[n] \mapsto G^n \times X$ as $[X/G]$. If $X = *$, we simply write BG. Then the ∞-category of G-objects is equivalent to the ∞-category $\mathsf{H}_{/BG}$.

Let $f\colon T \to [X/G]$ be a morphism. Pulling back along the projection map, we obtain a G-object P and a G-equivariant morphism $P \to X$, such that $[P/G] \to [X/G]$ is equivalent to f. A morphism $P \to T$ obtained this way is a *G-torsor* or *G-bundle.*

For an object V in H, let $\underline{\mathrm{Aut}}(V)$ be the internal automorphism ∞-group of V. This is constructed as follows. Up to size-issues which we will ignore, there is an object classifier $\widehat{\mathrm{Ob}(\mathsf{H})} \to \mathrm{Ob}(H)$, meaning that any object X of H fits in a Cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widehat{\mathrm{Ob}(\mathsf{H})} \arrow[d] \\ * \arrow[r] & \mathrm{Ob}(\mathsf{H}). \end{tikzcd}$$

Then $B\underline{\mathrm{Aut}}(V)$ is the ∞-image of the morphism $* \to \mathrm{Ob}(\mathsf{H})$ which classifies $V \to *$, where the ∞-image is by definition the colimit of the Čech nerve of $* \to \mathrm{Ob}(\mathsf{H})$. In particular, $* \to B\underline{\mathrm{Aut}}(V)$ is an effective epimorphism, and $\underline{\mathrm{Aut}}(V)$ is by definition the associated ∞-group.

For G an ∞-group in H, the *adjoint action* of G on itself is defined as the G-object associated to the projection $\mathscr{L}(BG) \to BG$. Here, $\mathscr{L}(BG)$ is the derived (free) loop stack, which is the self-intersection of the diagonal of BG, see §5.1. We write G_{ad} for this G-object, which has G as underlying object. By definition, this means that $[G_{\mathrm{ad}}/G] \simeq \mathscr{L}(BG)$.

Definition 3.4. For $\mathsf{H} = \mathsf{Stk}$, we introduce the following terminology. An ∞-group object, whose underlying object in Stk is a derived stack/algebraic stack/scheme, is called a *derived group stack/algebraic stack/scheme.* For a derived group scheme G, a *derived closed subgroup scheme* is a morphism $H \to G$ of derived group schemes which is a closed immersion on underlying derived schemes. Beware that, despite the name, a derived closed subgroup scheme $H \to G$ is in general not a monomorphism.

3.2. **Derived projective spectra.** Let $X \in \mathsf{Stk}$ be algebraic, let $\mathbb{M}$ be either $\mathbb{N}$ or $\mathbb{Z}$. As in the ungraded case, we have an ∞-category $\mathrm{QAlg}^{\mathbb{M}}(X)$ of $\mathbb{M}$-graded, quasi-coherent $\mathcal{O}_X$-algebras. The operations in the affine picture on $\mathbb{N}$-graded and $\mathbb{Z}$-graded algebras mentioned in §2.5 carry over to the global picture, for which we use the same notation.

Let $\mathcal{B} \in \mathrm{QAlg}^{\mathbb{N}}(X)$ be given. The *irrelevant ideal* $\mathcal{B}_+$ of $\mathcal{B}$ is the fiber of the map $\mathcal{B} \to \mathcal{B}_0$. Write $V(\mathcal{B}_+) \to \operatorname{Spec}\mathcal{B}$ for the closed immersion $\operatorname{Spec}(\mathcal{B}_0) \to \operatorname{Spec}\mathcal{B}$. The $\mathbb{N}$-grading on $\mathcal{B}$ induces a $\mathbb{G}_m$-action on $\operatorname{Spec}\mathcal{B}$, which restricts to a $\mathbb{G}_m$-action on $(\operatorname{Spec}\mathcal{B}) \setminus V(\mathcal{B}_+)$. Define the (relative) *projective spectrum* of $\mathcal{B}$ as the quotient

$$\operatorname{Proj}\mathcal{B} := [((\operatorname{Spec}\mathcal{B}) \setminus V(\mathcal{B}_+))/\mathbb{G}_m]$$

Projective spectra commute with base-change in the obvious way. It follows that $\operatorname{Proj}\mathcal{B}$ is always algebraic. Moreover, if $\mathcal{B}$ is generated in homogeneous degree 1 over $\mathcal{O}_X$, then for all derived schemes $f\colon T \to X$, the derived stack $(\operatorname{Proj}\mathcal{B}) \times_X T \simeq \operatorname{Proj}(f^*\mathcal{B})$ is a derived scheme as well. Here and in what follows, an A-algebra B is generated by a submodule $B_0 \subset B$ if the natural map $\mathrm{Sym}_A(B_0) \to B$ is surjective (on π_0).

3.3. **Weil restrictions.** Let $f\colon X \to Y$ be a morphism of stacks. Consider the pullback functor $f^*\colon \mathsf{Stk}_Y \to \mathsf{Stk}_X$. Suppose that f^* has a right adjoint f_*. Then, for $Z \to X$, we call f_*Z the *Weil restriction* of Z along f, written $\mathrm{Res}_f(Z)$.

Remark 3.5. Although in general the existence of the Weil restriction $\mathrm{Res}_f(-)$ is only a set-theoretic issue, we remark that it exists without any such issues by [SAG, Con. 19.1.2.3] when f is representable. Although Lurie's result is more general, and there are more general statements still, this version covers all cases that we will encounter.

Since colimits are universal in any ∞-topos (up to size issues), the ∞-category Stk is Cartesian closed. For a morphism of derived stacks $X \to Y$, the right adjoint of $X \times_Y (-)\colon \mathsf{Stk}_Y \to \mathsf{Stk}_Y$ is written $\underline{\mathrm{Map}}_{/Y}(X, -)$, and is called the *derived mapping stack*. Then $\mathrm{Res}_f(Z)$ is alternatively given by the stack

$$\mathrm{Res}_f(Z) = \underline{\mathrm{Map}}_{/Y}(X, Z) \times_{\underline{\mathrm{Map}}_{/Y}(X,X)} Y \tag{3.1}$$

Under certain finiteness conditions, the derived stack $\mathrm{Res}_f Z$ will be algebraic, see [HLP19, Thm. 5.1.1], [HKR22]. In terms of higher derived stacks, the algebraicity of $\mathrm{Res}_f Z$ is controlled by the algebraicity of $Z \to X$ and the Tor-amplitude of $X \to Y$. Instead of giving a general statement, we will indicate the algebraicity of the Weil restriction in particular cases when relevant.

3.4. **Derived Rees algebras.** Endow $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t^{-1}]$ with the $\mathbb{G}_m$-action which corresponds to declaring t^{-1} to be of homogeneous degree -1. For $X \in \mathsf{Stk}$ algebraic, consider the map $\zeta_X : B\mathbb{G}_m \times X \to [\mathbb{A}^1/\mathbb{G}_m] \times X$ induced by the zero section. Then we can Weil-restrict along ζ_X, i.e., we have a right adjoint $\mathrm{Res}_\zeta : \mathsf{Stk}_{B\mathbb{G}_m \times X} \to \mathsf{Stk}_{[\mathbb{A}^1/\mathbb{G}_m]\times X}$ to the pullback functor.

Let now $Z \to X$ be a morphism of derived stacks, with X still algebraic. Define the *deformation space* of $Z \to X$ as the Weil restriction $\mathscr{D}_{Z/X} :=$ $\mathrm{Res}_{\zeta_X}(Z \times B\mathbb{G}_m)$ of $Z \times B\mathbb{G}_m$ along ζ_X. Write $D_{Z/X}$ for the pullback of $\mathscr{D}_{Z/X}$ along $\mathbb{A}^1 \times X \to [\mathbb{A}^1/\mathbb{G}_m] \times X$. In [HKR22] it is shown that the deformation space is not only functorial in Z, but also in the morphism $Z \to X$. The resulting functor $\mathscr{D}_{(-)/(-)}$ is a right adjoint, hence in particular is stable under base-change. Consequently, the same is true for $D_{(-)/(-)}$.

From here on, let $Z \to X$ be a closed immersion of derived algebraic stacks. Then $D_{Z/X} \to \mathbb{A}^1 \times X$ is affine by [Hek21], and $D_{Z/X}$ carries a $\mathbb{G}_m$-action such that $[D_{Z/X}/\mathbb{G}_m] \simeq \mathscr{D}_{Z/X}$. We define the (derived) *extended Rees algebra* of Z over X as the $\mathbb{Z}$-graded, quasi-coherent $\mathcal{O}_X[t^{-1}]$-algebra $\mathcal{R}^{\mathrm{ext}}_{Z/X}$ such that $D_{Z/X} = \operatorname{Spec} \mathcal{R}^{\mathrm{ext}}_{Z/X}$. We let $\mathcal{R}_{Z/X} := (\mathcal{R}^{\mathrm{ext}}_{Z/X})_{\geq 0}$ be the *Rees algebra* of Z over X. In the affine case, say with $Z = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, we write $R^{\mathrm{ext}}_{B/A}, R_{B/A}$ for the algebras corresponding to the quasi-coherent algebras $\mathcal{R}^{\mathrm{ext}}_{Z/X}, \mathcal{R}_{Z/X}$.

The main properties of this construction are as follows. The (extended) Rees algebra is stable under base-change in the obvious way. Since $Z \to X$ is affine, we can write $Z = \operatorname{Spec} \mathcal{B}$ for some quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$. We can then recover $\mathcal{B}$ from the Rees algebra as $(\mathcal{R}^{\mathrm{ext}}_{Z/X}/(t^{-1}))_0 \simeq \mathcal{B}$. In fact, the pullback of $D_{Z/X} \to \mathbb{A}^1 \times X$ to $\{0\} \times X$ gives the normal bundle $\mathbb{N}_{Z/X}$, living over $\{0\} \times X$ via the map $Z \to \{0\} \times X$.

Example 3.6. Let $A \in \mathsf{Alg}$, let $\sigma_1, \dots, \sigma_k$ be cycles $\sigma_i \in \pi_{n_i}(A)$, and let B be the finite quotient $A/(\sigma_1, \dots, \sigma_k)$. Then

$$R^{\mathrm{ext}}_{B/A} \simeq \frac{A[t^{-1}, u_1, \dots, u_n]}{(u_1 t^{-1} - f_1, \dots, u_n t^{-1} - f_n)}$$

where the u_i are free in homogeneous degree 1 and homological degree n_i.

3.5. **Derived blow-ups and virtual Cartier divisors.** We continue with the closed immersion $Z \to X$ of derived algebraic stacks. We define the *derived blow-up* of X in Z as the projective spectrum of $\mathcal{R}_{Z/X}$ over X, written $\mathrm{Bl}_Z X$.

The basic properties here are as follows. The construction is stable under arbitrary base-change in the obvious way, and if Z, X are derived schemes, then so is $\mathrm{Bl}_Z X$. It follows that, in general, $\mathrm{Bl}_Z X$ is a derived algebraic stack. The structure map $\mathrm{Bl}_Z X \to X$ is always an equivalence outside Z. Over Z, we have the *exceptional divisor* $E_Z X$, which is the projective normal bundle $\mathbb{P}(\mathcal{N}_{Z/X}) = [(\mathbb{N}_{Z/X} \setminus 0)/\mathbb{G}_m]$, and lives over $Z \to X$ as a virtual Cartier divisor (see below). We recover the classical blow-up of X_{cl} in Z_{cl} as the schematic closure of $(\mathrm{Bl}_Z X)_{\mathrm{cl}} \setminus (E_Z X)_{\mathrm{cl}}$ inside $(\mathrm{Bl}_Z X)_{\mathrm{cl}}$.

Let $j\colon D \to T$ be a closed immersion of derived algebraic stacks. Then j is called *quasi-smooth of virtual codimension* n if, for any derived scheme $T' \to T$, there are locally defined sections $f_1, \dots, f_n \in \mathcal{O}_{T'}$ such that, Zariski locally on T', the pullback $D \times_T T'$ is the derived vanishing locus $V(f_1, \dots, f_n) = \mathrm{Spec}(\mathcal{O}_{T'}/(f_1, \dots, f_n))$. A *virtual Cartier divisor* is a quasi-smooth closed immersion of virtual codimension 1.

Example 3.7. The map $\zeta\colon B\mathbb{G}_m \to [\mathbb{A}^1/\mathbb{G}_m]$ is a virtual Cartier divisor. In fact, this is the universal example: for any virtual Cartier divisor $j\colon D \to T$ there is a map $g\colon T \to [\mathbb{A}^1/\mathbb{G}_m]$ which classifies j in the sense that j is the pullback of ζ along g.

Definition 3.8. Let $Z \to X$ be a morphism of derived algebraic stacks, and T a derived algebraic stack. A *virtual Cartier divisor over $Z \to X$ (on T)* is a T-point of $\mathscr{D}_{Z/X}$.

The universality of ζ implies that a virtual Cartier divisor over $Z \to X$ is a commutative diagram of the form

$$\begin{array}{ccc} D & \longrightarrow & T \\ \downarrow{\scriptstyle g} & & \downarrow \\ Z & \longrightarrow & X \end{array} \tag{3.2}$$

where $D \to T$ is a virtual Cartier divisor.

Definition 3.9. If $Z \to X$ is a closed immersion, then a virtual Cartier divisor over $Z \to X$ classified by the diagram 3.2 is *strict* when the underlying classical diagram is Cartesian, and the induced homomorphism $g^*\mathcal{N}_{Z/X} \to \mathcal{N}_{D/T}$ is surjective on π_0. (This terminology deviates from [KR18], but agrees with [HKR22]).

The derived blow-up represents strict virtual Cartier divisors over $Z \to X$ in the sense that $\mathrm{Bl}_Z X(T)$ is equivalent to the space of strict virtual Cartier divisors over $Z \to X$ on T, with the exceptional divisor as the universal example.

3.6. **Properties and formulas of derived blow-ups.** We now prove and collect a few additional properties of derived blow-ups, which we will use in Section 6.

As a consequence of the functorial description of the derived blow-up, we have the following proposition, which shows that the classical truncation $(\mathrm{Bl}_Z U)_{\mathrm{cl}}$ depends on $Z_{\leq 1} := \pi_{\leq 1} Z$, hence in particular is invariant under attaching n-cells to $\mathcal{O}_Z$ for $n \geq 2$.

Proposition 3.10. *Suppose that $Z' \to Z$ is a closed embedding such that $Z'_{\leq 1} \to Z_{\leq 1}$ is an equivalence. Then the natural morphism $\mathrm{Bl}_{Z'} U \to \mathrm{Bl}_Z U$ induces an isomorphism on classical truncations.*

Proof. The classical truncations $(\mathrm{Bl}_{Z'} U)_{\mathrm{cl}}$ and $(\mathrm{Bl}_Z U)_{\mathrm{cl}}$ are determined by their T-points, where T is a classical affine scheme, and a virtual Cartier divisor $D \to T$ is locally (on T) the vanishing locus of a section $f \in \mathcal{O}_T$. In particular, since T is classical, the truncation $D_{\leq 1} \to D$ is an equivalence. Since also $Z'_{\leq 1} \simeq Z_{\leq 1}$, any map $D \to Z$ factors uniquely through Z'. □

We also have the following simple proposition regarding classical truncations of derived blow-ups and pullback squares.

Proposition 3.11. *Suppose we are given a Cartesian diagram*

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & U' \arrow[d] \\ Z \arrow[r] & U \end{tikzcd}$$

of derived schemes, such that $U'_{\mathrm{cl}} \to U_{\mathrm{cl}}$ is an isomorphism. Then the canonical map $(D_{Z'/U'})_{\mathrm{cl}} \longrightarrow (D_{Z/U})_{\mathrm{cl}}$ is an isomorphism.

Proof. Since the deformation space is stable under pullbacks, we have

$$(D_{Z'/U'})_{\mathrm{cl}} \simeq (D_{Z/U} \times_U U')_{\mathrm{cl}} \simeq (D_{Z/U})_{\mathrm{cl}}$$

where the last equivalence follows since $(-)_{\mathrm{cl}}$ commutes with pullbacks, and since $U_{\mathrm{cl}} \simeq U'_{\mathrm{cl}}$ by assumption. □

In order to perform explicit computations, we will also need the following lemma. Note that $\mathscr{D}_{U/U} \simeq Y \times [\mathbb{A}^1/\mathbb{G}_m]$, hence $D_{U/U} \simeq Y \times \mathbb{A}^1$.

Lemma 3.12. *Let a sequence $Z \to U \to V$ of derived stacks be given. Then the natural map*

$$\mathscr{D}_{Z/U} \to \mathscr{D}_{U/U} \times_{\mathscr{D}_{U/V}} \mathscr{D}_{Z/V}$$

induced by naturality of $\mathscr{D}_{(-)/(-)}$, is an equivalence. Consequently, the same holds true if we replace $\mathscr{D}_{(-)/(-)}$ by $D_{(-)/(-)}$ and $\mathsf{Stk}_{[\mathbb{A}^1/\mathbb{G}_m]}$ by $\mathsf{Stk}^{\mathbb{G}_m}_{\mathbb{A}^1}$.

Proof. This will appear in [HKR22]. □

Suppose now that A is any algebra and we are given a sequence of closed embeddings $Z \to W \to U \to V = \operatorname{Spec} A$, corresponding to

$$A \to B \to C \to D$$

where $B = A/(f_1, ..., f_n)$ for certain $f_i \in \pi_0 A$, and $C = B/(\sigma_1, ..., \sigma_k)$ for certain $\sigma_i \in \pi_{d_i} B$, and $D = A/(g_1, ..., g_m)$ for certain $g_i \in \pi_0 A$.

We use multi-index notation, so that $B = A/(\underline{f})$ and so on. From the theory of Rees algebras, we have a diagram

$$\begin{tikzcd} (R^{\mathrm{ext}}_{U/V})_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[d] \\ (R^{\mathrm{ext}}_{Z/V})_1 \arrow[r] & A \arrow[r] & D, \end{tikzcd}$$

where the rows are fiber sequences. Write $R^{\text{ext}}_{U/V} = A[t^{-1}, \underline{v}]/(t^{-1}\underline{v} - \underline{f})$, and $R^{\text{ext}}_{Z/V} = A[t^{-1}, \underline{w}]/(t^{-1}\underline{w} - \underline{g})$, and p for the composition $\pi_0 A[t^{-1}, \underline{v}] \to \pi_0 R^{\text{ext}}_{U/V} \to \pi_0 R^{\text{ext}}_{Z/V}$. Here, $\underline{v} = (v_1, \ldots, v_n)$ and $\underline{w} = (w_1, \ldots, w_m)$ are free in homological degree 0 and homogeneous degree 1.

For each i, let v'_i be the image of v_i under p. Since p sends $t^{-1}v_i - f_i$ to zero, we get $\lambda_{ji} \in \pi_0 A[t^{-1}, \underline{w}]$ such that

$$t^{-1}v'_i - f_i = \sum\nolimits_j \lambda_{ji}(t^{-1}w_j - g_j)$$

holds in $\pi_0 A[t^{-1}, \underline{w}]$. Putting $t^{-1} = 0$ gives us that $f_i = \sum_j \lambda_{ji} g_j$, in particular that $\lambda_{ji} \in \pi_0 A$. Putting $t^{-1} = 1$ then gives $v'_i = \sum_j \lambda_{ji} w_j$. It follows that $R^{\text{ext}}_{U/V} \to R^{\text{ext}}_{Z/V}$ sends v_i to $\sum_j \lambda_{ij} g_j$, in the sense that precomposing with $A[t^{-1}, \underline{v}] \to R^{\text{ext}}_{U/V}$ gives the unique map $A[t^{-1}, \underline{v}] \to R^{\text{ext}}_{Z/V}$ of $A[t^{-1}]$-algebras that sends v_i to $\sum_j \lambda_{ij} g_j$.

Write $R^{\text{ext}}_{W/U} = B[t^{-1}\underline{u}]/(t^{-1}\underline{u} - \underline{\sigma})$ where $\underline{u} = (u_1, \ldots, u_k)$ are free in homogeneous degree 1 and homological degrees $d_1, \ldots, d_k$. Then, with the same argument as before, we get elements $(\tau_1, \ldots, \tau_k)$ in $\pi_* R^{\text{ext}}_{Z/U}$, with τ_i in homological degree d_i and homogeneous degree 1 such that $\pi_* R^{\text{ext}}_{W/U} \to \pi_* R^{\text{ext}}_{Z/U}$ sends u_i to τ_i.

Proposition 3.13. *We have*

$$R^{\text{ext}}_{Z/U} \simeq \frac{A[t^{-1}, \underline{w}]}{(t^{-1}\underline{w} - \underline{g}, \sum_j \lambda_{1j} w_j, \ldots, \sum_j \lambda_{nj} w_j)} \tag{3.3}$$

where $\underline{w} = (w_1, \ldots, w_m)$ are in homological degree 0 and homogeneous degree 1. Likewise, we have

$$R^{\text{ext}}_{Z/W} \simeq R^{\text{ext}}_{Z/U}/(\underline{\tau}) \tag{3.4}$$

where $\underline{\tau} = (\tau_1, \ldots, \tau_k)$ are the elements with τ_i in homological degree d_i and homogeneous degree 1 described above.

Remark 3.14. As the proof will make clear, there is only a contractible space of choices involved here, since the equivalence comes about from a *description* of canonically given maps.

Proof. By Lemma 3.12, we have $R^{\text{ext}}_{Z/U} \simeq R^{\text{ext}}_{U/U} \otimes_{R^{\text{ext}}_{U/V}} R^{\text{ext}}_{Z/V}$. We thus get the following diagram, where the transition maps in the bottom right corner are the canonical maps induced by functoriality of $R^{\text{ext}}_{(-)/(-)}$.

$$\begin{tikzcd}
\mathbb{Z}[\underline{F}] \arrow[r, "\underline{F}\mapsto 0"] \arrow[d, "\underline{F}\mapsto t^{-1}\underline{v}-\underline{f}"] & \mathbb{Z} \arrow[d] & \\
A[t^{-1}, \underline{v}] \arrow[r] \arrow[d, "\underline{v}\mapsto 0"] & \frac{A[t^{-1},\underline{v}]}{(t^{-1}\underline{v}-\underline{f})} \arrow[r, "v_i\mapsto\sum_j \lambda_{ij}w_j"] \arrow[d] & \frac{A[t^{-1},\underline{w}]}{(t^{-1}\underline{w}-\underline{g})} \arrow[d] \\
A[t^{-1}] \arrow[r] & B[t^{-1}] \arrow[r] & R^{\text{ext}}_{Z/U}
\end{tikzcd}$$

where all squares are pushouts, and $\underline{v} = (v_1, \ldots, v_n)$ are free in homogeneous degree 1 and homological degree 0, and likewise for $\underline{F} = (F_1, \ldots, F_n)$. The first claim follows.

For the second claim, we use $R^{\mathrm{ext}}_{Z/W} \simeq R^{\mathrm{ext}}_{W/W} \otimes_{R^{\mathrm{ext}}_{W/U}} R^{\mathrm{ext}}_{Z/U}$ to get the diagram

$$\begin{tikzcd}
\mathbb{Z}[\underline{U}] \arrow[r, "\underline{U}\mapsto \underline{u}t^{-1}-\underline{\sigma}"] \arrow[d, "\underline{U}\mapsto 0"] & B[t^{-1}, \underline{u}] \arrow[r, "\underline{u}\mapsto 0"] \arrow[d] & B[t^{-1}] \arrow[d] \\
\mathbb{Z} \arrow[r] & \frac{B[t^{-1}, \underline{u}]}{(\underline{u}t^{-1}-\underline{\sigma})} \arrow[r] \arrow[d, "\underline{u}\mapsto \underline{\tau}"] & C[t^{-1}] \arrow[d] \\
 & R^{\mathrm{ext}}_{Z/U} \arrow[r] & R^{\mathrm{ext}}_{Z/W}
\end{tikzcd}$$

where $\underline{U} = (U_1, \ldots, U_k)$ are free in homogeneous degree 1 and homological degrees $d_1, \ldots, d_k$. Since all squares are pushouts, the second claim follows. □

4. Equivariant derived algebraic geometry

A main ingredient in the canonical reduction of stabilizers algorithm for algebraic stacks from [ER21] is the locus $X^{\max}$ of points with maximal-dimensional stabilizers, for a given algebraic stack X. This locus is a closed substack of X in the case that X admits a good moduli space, as shown in [ER21, Prop. B.4].

In the next section, §5, we give a derived analogue of $X^{\max}$ in the case that X is a derived algebraic stack for which X_{cl} admits a good moduli space. In order to do this, in this section we give a treatment of several elements of equivariant geometry and study fixed loci in the derived setting. This allows us to obtain equivariant versions of the results of the previous section, including equivariant derived blow-ups. We note that there is some overlap between what we present here and what is presented in [AKL$^+$22, §A.3].

4.1. **Equivariant rings, modules and stacks.** Fix a reductive algebraic group G over $\mathbb{C}$. By definition, G is a linear algebraic group, hence a smooth closed subscheme of some GL_n (smooth by Cartier's theorem). A *classical G-module V* will be an algebraic representation of G on a vector space V, i.e., a morphism of group schemes $G \to \mathrm{GL}(V)$. Write Rep^G for the category of finite dimensional G-modules V. Recall, since algebraic representations are rational, every $M \in \mathsf{Rep}^G$ splits into a direct sum of simple modules.

Definition 4.1. Let the symmetric monoidal ∞-category of *G-modules*, written Mod^G, be the stabilization of $\mathsf{P}_\Sigma(\mathsf{Rep}_G)$ with symmetric monoidal structure induced from Rep_G, and let Alg^G be the ∞-category of connective $\mathbb{E}_\infty$-algebras in Mod^G.

Recall that the heart $\mathrm{QCoh}(BG)^\heartsuit$ is the category of classical G-modules. In particular, we have a canonical inclusion $\mathsf{Rep}_R \to \mathrm{QCoh}(BG)$. Since $\mathrm{QCoh}(BG)_{\geq 0}$ is generated under colimits by the image of $\mathsf{Rep}_G \to \mathrm{QCoh}(BG)$, the canonical map $\mathsf{P}_\Sigma(\mathsf{Rep}_G) \to \mathrm{QCoh}(BG)_{\geq 0}$ is an equivalence, hence $\mathsf{Mod}^G \simeq \mathrm{QCoh}(BG)$. We endow Mod^G with the canonical t-structure induced from $\mathrm{QCoh}(BG)$.

Since the ∞-category $\mathsf{Mod}^G_{\geq 0}$ of connective G-modules is presentable and the tensor product of Mod^G commutes with colimits in each variable separately, the forgetful functor $\mathsf{Alg}^G \to \mathsf{Mod}^G_{\geq 0}$ admits a left adjoint, written as

$$\mathrm{LSym}^G\colon \mathsf{Mod}^G_{\geq 0} \to \mathsf{Alg}^G.$$

Let Poly^G be the full subcategory of Alg^G spanned by objects of the form $\mathrm{LSym}^G(M)$, for $M \in \mathsf{Rep}^G$. Then, since we are in characteristic zero and the G-actions we consider are rational, Alg^G is freely generated under sifted colimits under Poly^G. By definition, this implies the following.

Lemma 4.2. *Let $B \in \mathsf{Alg}^G$. Then the canonical map*

$$\mathrm{colim}_{B_\alpha \to B}\, B_\alpha \to B$$

is an equivalence, where the indexing category consists of all equivariant maps $B_\alpha \to B$ with $B_\alpha \in \mathsf{Poly}^G$.

In the terminology from [Rak20], the ∞-category Mod^G is a derived algebraic context, and Alg^G is then the associated ∞-category of connective derived rings.

The forgetful functor $\mathsf{Mod}^G \to \mathsf{Mod}$ is symmetric monoidal, which gives us a canonical forgetful functor $\mathsf{Alg}^G \to \mathsf{Alg}$. Also, the functor $\mathsf{Mod} \to \mathsf{Mod}^G$ which endows a module with trivial G-action preserves colimits. We write the right adjoint of this functor as $(-)^{\mathrm{fix}}$. Note that $(-)^{\mathrm{fix}}$ corresponds to taking the classical fixed part on Rep^G.

Definition 4.3. Write Aff^G for the ∞-category $(\mathsf{Alg}^G)^{\mathrm{op}}$. Endow Aff^G with the *étale topology* by declaring a family $\{U_i \to X\}$ in Aff^G to be a cover if the underlying family of derived affine schemes is an étale cover. Write Stk^G for the ∞-category of derived stacks on Aff^G with respect to the étale topology, and $\mathrm{Spec}^G\colon (\mathsf{Alg}^G)^{\mathrm{op}} \to \mathsf{Stk}^G$ for the Yoneda embedding.

For a derived stack X, write $\mathsf{Aff}(X)$ for the ∞-category of stacks which are affine over X, and $\mathrm{QAlg}(X)$ for the ∞-category of quasi-coherent $\mathcal{O}_X$-algebras, so that we have an equivalence

$$\mathrm{Spec}(-)\colon \mathrm{QAlg}(X)^{\mathrm{op}} \to \mathsf{Aff}(X).$$

Since we are in characteristic zero, $\mathrm{QAlg}(X)$ is also the ∞-category of connective $\mathbb{E}_\infty$-algebras in $\mathrm{QCoh}(X)$. The case $X = BG$ therefore gives us

$$\mathsf{Aff}^G \simeq \mathsf{Aff}(BG)$$

Recall that pulling back along the quotient map $\varphi\colon * \to BG$ induces an equivalence between Stk_{BG} and the ∞-category of stacks with a G-action. Since φ is affine, this implies that Aff^G is also the ∞-category of affine derived schemes with a G-action.

Proposition 4.4. *The ∞-category Stk^G is equivalent to Stk_{BG}.*

Proof. Let T be any derived stack. Since T has affine diagonal, the ∞-category $\mathsf{Aff}_{/T}$ of derived affine schemes over T is a full subcategory of $\mathsf{Aff}(T)$. Composition with the inclusion $f\colon \mathsf{Aff}_{/T} \to \mathsf{Aff}(T)$ and left Kan extension gives an adjunction

$$f_! : \mathsf{Stk}(\mathsf{Aff}_{/T}) \rightleftarrows \mathsf{Stk}(\mathsf{Aff}(T)) : f^*$$

Since f is fully faithful, so is $f_!$. Moreover, since $\mathsf{Stk}_T \simeq \mathsf{Stk}(\mathsf{Aff}_{/T})$, the image of $f_!$ contains all stacks over T. Since $\mathsf{Stk}(\mathsf{Aff}(T))$ is generated under colimits by stacks affine over T, it follows that $f_!$ is essentially surjective. Hence $\mathsf{Stk}_{/T} \simeq \mathsf{Stk}(\mathsf{Aff}(T))$, and putting $T = BG$ gives us what we want. $\square$

Although we will use the equivalence $\mathsf{Stk}^G \simeq \mathsf{Stk}_{BG}$ seamlessly, we will not conflate a derived stack X over BG with the corresponding derived G-stack $P \coloneqq X \times_{BG} \{*\}$. In particular, if X is endowed with a G-action as derived stack over BG, then this contains no information regarding the action of P.

Notation 4.5. Let G, G' be reductive groups. The equivalences $\mathsf{Stk}^G \simeq \mathsf{Stk}_{BG}$ and $\mathsf{Stk}^{G'} \simeq \mathsf{Stk}_{BG'}$ induce equivalences on functor-∞-categories, which we write as

$$B : \mathrm{Fun}(\mathsf{Stk}^G, \mathsf{Stk}^{G'}) \leftrightarrows \mathrm{Fun}(\mathsf{Stk}_{BG}, \mathsf{Stk}_{BG'}) : \Omega$$

Remark 4.6. Consider the map $\psi\colon BG \to *$. Then the functor $\mathsf{Stk} \to \mathsf{Stk}_{BG}$ given by pulling back along ψ is right adjoint to the functor $\mathsf{Stk}_{BG} \to \mathsf{Stk}$ given by composing with ψ. Under the equivalence $\mathsf{Stk}_{BG} \simeq \mathsf{Stk}^G$, this corresponds to the adjunction

$$[-/G] : \mathsf{Stk}^G \rightleftarrows \mathsf{Stk} : \iota$$

where ι endows a derived stack with the trivial action.

We also have an adjunction

$$(-) \times G : \mathsf{Stk} \leftrightarrows \mathsf{Stk}^G : U$$

induced by $\varphi\colon * \to BG$, where U is the forgetful functor, and the G-action on $Y \times G$ is given by translation, for $Y \in \mathsf{Stk}$. Essentially, this is because any G-equivariant map $Y \times G \to X$ is determined by its restriction to $Y \times \{e\} \to X$.

Let $B \in \mathsf{Alg}^G$. Define the ∞-category Mod_B^G of (G, B)-*modules* as the ∞-category of B-modules in Mod^G, with canonical t-structure. Write Alg_B^G for the slice category $(\mathsf{Alg}^G)_{B/}$. As in the absolute case, we have an adjunction

$$\mathrm{LSym}_B^G : \mathsf{Mod}_B^G \leftrightarrows \mathsf{Alg}_B^G : U$$

where U is the forgetful functor. Let Rep_B^G be the full-∞-subcategory of Mod_B^G spanned by objects of the form $B \otimes M$, where $M \in \mathsf{Rep}^G$. As shown in [BH22], Mod_B^G is again a derived algebraic context, and Rep_B^G generates $(\mathsf{Mod}_B^G)_{\geq 0}$ under sifted colimits.

As in the non-equivariant case, we can globalize this picture via a right Kan extension to a functor

$$\mathrm{QCoh}^G(-)\colon (\mathsf{Stk}^G)^{\mathrm{op}} \to \mathsf{Cat}$$

such that $\mathrm{QCoh}^G(X)$ is stable, presentable and symmetric monoidal for X algebraic. Moreover, for $f\colon X \to Y$ an equivariant map of derived algebraic stacks, we have an equivariant pullback functor

$$(f^G)^*\colon\ \mathrm{QCoh}^G(Y) \to \mathrm{QCoh}^G(X) \tag{4.1}$$

which is symmetric monoidal and a left adjoint.

With the same procedure, we get a functor $\mathrm{QAlg}^G(-)$. Now for $X \in \mathsf{Stk}^G$ and $\mathcal{A} \in \mathrm{QAlg}^G(X)$, it holds that $\mathcal{A}$ is an $\mathbb{E}_\infty$-algebra in $\mathrm{QCoh}^G(X)$. We can thus define the ∞-category $\mathrm{QCoh}^G_{\mathcal{A}}(X)$ of $(G, \mathcal{A})$-modules as the ∞-category of $\mathcal{A}$-modules in $\mathrm{QCoh}^G(X)$. The subcategory of $\mathrm{QCoh}^G_{\mathcal{A}}(X)$ spanned by modules with trivial G-action is written $\mathrm{QCoh}_{\mathcal{A}}(X)$, which is equivalent to the ∞-category of $\mathcal{A}$-modules in $\mathrm{QCoh}(X)$.

4.2. **Restriction and co-induction.** Suppose that a reductive algebraic group G acts trivially on $A \in \mathsf{Alg}$. Then Mod^G_A is equivalent to the ∞-category of A-modules endowed with a G-action, i.e., we have an equivalence

$$\mathsf{Mod}^G_A \simeq \mathsf{Mod}^G \otimes_{\mathsf{Mod}} \mathsf{Mod}_A \simeq \mathrm{Fun}_\sigma((\mathsf{Rep}^G)^{\mathrm{op}}, \mathsf{Mod}_A) \tag{4.2}$$

where $\mathrm{Fun}_\sigma((\mathsf{Rep}^G)^{\mathrm{op}}, \mathsf{Mod}_A)$ is the ∞-category of functors $(\mathsf{Rep}^G)^{\mathrm{op}} \to \mathsf{Mod}_A$ that send finite coproducts in Rep^G to products in Mod_A, and $(-)\otimes(-)$ is the Lurie tensor product, see [Rak20, Rem. 2.2.8].

Let now $f\colon G \to G'$ be a homomorphism of reductive algebraic groups, and let G' also act trivially on A. Write the induced morphism on stacks as $\varphi\colon \operatorname{Spec} A \times BG \to \operatorname{Spec} A \times BG'$, and the adjunction $\mathrm{QCoh}(\operatorname{Spec} A \times BG') \leftrightarrows \mathrm{QCoh}(\operatorname{Spec} A \times BG)$ induced by φ as

$$\mathrm{Res}_\varphi : \mathsf{Mod}^{G'}_A \leftrightarrows \mathsf{Mod}^G_A : \mathrm{cInd}_\varphi$$

We call Res_φ the *restriction* functor, and cInd_φ the *co-induction* functor.

Recall (see, e.g., [GR17, §3.2]), that Res_φ is symmetric monoidal and that cInd_φ is right-lax symmetric monoidal. Since BG' has affine and flat diagonal, $BG \to BG'$ is flat and cohomologically affine. It follows that the functors Res_φ and cInd_φ are t-exact. The adjunction therefore lifts to the level of algebras, which we again write as

$$\mathrm{Res}_\varphi : \mathsf{Alg}^{G'}_A \leftrightarrows \mathsf{Alg}^G_A : \mathrm{cInd}_\varphi$$

Notation 4.7. Consider the maps $\varphi\colon\ \operatorname{Spec} A \to \operatorname{Spec} A \times BG'$ and $\psi\colon\ \operatorname{Spec} A \times BG \to \operatorname{Spec} A$. Then the functor $\mathrm{Res}_\varphi\colon \mathsf{Mod}^{G'}_A \to \mathsf{Mod}_A$ is forgetting the G'-action, which we will write as $(-)^u$; the functor $\mathrm{Res}_\psi\colon \mathsf{Mod}_A \to \mathsf{Mod}^G_A$ endows an A-module with trivial G-action, which we will write as $(-)^{\mathrm{tv}}$; and $\mathrm{cInd}_\psi\colon \mathsf{Mod}^G_A \to \mathsf{Mod}_A$ sends an A-module with G-action M to the G-fixed points M^{fix}. Again, we use the same notation for the induced functors on algebras.

Globalizing this via right Kan extensions gives us an adjunction

$$i : \mathrm{QCoh}(X) \leftrightarrows \mathrm{QCoh}^G(X) : (-)^{\mathrm{fix}} \tag{4.3}$$

for any stack X with trivial G-action.

Definition 4.8. Let $X \in \mathsf{Stk}$ algebraic with trivial G-action and $\mathcal{F} \in \mathrm{QCoh}^G(X)$ be given. Consider the counit $\mathcal{F}^{\mathrm{fix}} \to \mathcal{F}$ of the adjunction $i \dashv (-)^{\mathrm{fix}}$. The *moving part* of $\mathcal{F}$ is the cofiber of the counit, written $\mathcal{F} \to \mathcal{F}^{\mathrm{mv}}$. We write $\mathrm{QCoh}^{\mathrm{mv}}(X)$ for the essential image of the functor $(-)^{\mathrm{mv}} \colon \mathrm{QCoh}^G(X) \to \mathrm{QCoh}^G(X)$.

Recall that a sequence of colimit-preserving functors

$$\mathsf{C} \xrightarrow{i} \mathsf{D} \xrightarrow{g} \mathsf{E}$$

between presentable stable ∞-categories is *exact* if $gi \simeq 0$, i is fully faithful, and E is the Verdier quotient D/C, i.e., it is the localization of D at the morphisms for which the cofiber is in C. Such a sequence is *split-exact* if the right adjoints $i \dashv f$ and $g \dashv j$, which exist by the adjoint functor theorem, are such that $fi \simeq \mathrm{id}_{\mathsf{C}}$ and $gj \simeq \mathrm{id}_{\mathsf{E}}$.

Remark 4.9. The proof of the third point of Proposition 4.10 is inspired by [BR19, App. A], and uses the following facts on (split) exact sequences that can be found in the detailed exposition [CDH+20, App. A] on the matter.

A sequence $\mathsf{C} \xrightarrow{i} \mathsf{D} \xrightarrow{g} \mathsf{E}$ with adjoints j, f as above is exact if and only if i is a fully faithful functor with essential image spanned by the objects $M \in \mathsf{D}$ such that $gM \simeq 0$. If the sequence is moreover split, then the sequence $\mathsf{E} \xrightarrow{j} \mathsf{D} \xrightarrow{f} \mathsf{C}$ is also a split exact sequence. In this case, the square

$$\begin{tikzcd} M \arrow[r] \arrow[d] & jg(M) \arrow[d] \\ if(M) \arrow[r] & 0 \end{tikzcd}$$

induced by the units of $g \dashv j$ and $f \dashv i$, is Cartesian, for any $M \in \mathsf{D}$.

Proposition 4.10. *Let X be a derived algebraic stack with a G-action and $\varphi_X \colon X \to [X/G]$ the projection map. Consider the equivariant pullback functor*

$$(\varphi_X^G)^* \colon \mathrm{QCoh}^G([X/G]) \to \mathrm{QCoh}^G(X).$$

as displayed in 4.1, where the G-action on $[X/G]$ is trivial.

(i) *Composing $(\varphi_X^G)^*$ with $i_{[X/G]} \colon \mathrm{QCoh}([X/G]) \to \mathrm{QCoh}^G([X/G])$ from (4.3) induces an equivalence*

$$\mathrm{QCoh}([X/G]) \simeq \mathrm{QCoh}^G(X).$$

(ii) *Suppose that the G-action on X is trivial and let $\psi_X \colon [X/G] \to X$ be the structure map. Composing the adjunction $\psi_X^* \dashv \psi_{X*}$ with the equivalence from (i) yields the adjunction*

$$i : \mathrm{QCoh}(X) \leftrightarrows \mathrm{QCoh}^G(X) : (-)^{\mathrm{fix}}$$

from (4.3).

(iii) *Suppose that the G-action on X is trivial. Then the sequence*

$$\mathrm{QCoh}(X) \xrightarrow{\ i\ } \mathrm{QCoh}^G(X) \xrightarrow{(-)^{\mathrm{mv}}} \mathrm{QCoh}^{\mathrm{mv}}(X)$$

is split exact.

Proof. (i). By descent, this is a local question, so we may assume that X is affine, say $X = \operatorname{Spec} A$. Let $\mathcal{A} \in \mathrm{QAlg}(BG)$ correspond to $A \in \mathsf{Alg}^G$ under the equivalence $\mathsf{Alg}^G \simeq \mathrm{QAlg}(BG)$, so that $[T/G] = \operatorname{Spec} \mathcal{A}$ and $\mathrm{QCoh}([X/G]) \simeq \mathrm{QCoh}_{\mathcal{A}}(BG)$. Now the G-actions on BG, on $[X/G]$, and on $\mathcal{A}$ are all trivial. Hence, with the same argument as in (4.2), we also have

$$\mathrm{QCoh}^G([X/G]) \simeq \mathrm{QCoh}^G_{\mathcal{A}}(BG) \simeq \mathrm{QCoh}^G(BG) \otimes_{\mathrm{QCoh}(BG)} \mathrm{QCoh}_{\mathcal{A}}(BG)$$

Moreover, $\mathrm{QCoh}^G(X) \simeq \mathsf{Mod}^G_A \simeq \mathrm{QCoh}_{\mathcal{A}}(BG)$ holds by definition of Mod^G_A. The composition $(\varphi^G_X)^* \circ i_{[X/G]}$ is then equivalent to the composition

$$\mathrm{QCoh}_{\mathcal{A}}(BG) \to \mathrm{QCoh}^G(BG) \otimes_{\mathrm{QCoh}(BG)} \mathrm{QCoh}_{\mathcal{A}}(BG) \to \mathrm{QCoh}_{\mathcal{A}}(BG)$$

where the first arrow sends $M \in \mathrm{QCoh}_{\mathcal{A}}(BG)$ to the external tensor product $\mathcal{O}_{BG} \boxtimes M$, where $\mathcal{O}_{BG}$ has trivial G-action, and the second arrow sends $N \boxtimes K \in \mathrm{QCoh}^G(BG) \otimes_{\mathrm{QCoh}(BG)} \mathrm{QCoh}_{\mathcal{A}}(BG)$ to $N^u \otimes K$. The claim follows.

(ii). It suffices to show that $i \simeq (\varphi^G_X)^* \circ i_{[X/G]} \circ \psi^*_X$. Since the action on X is now trivial, we have $\mathrm{QCoh}([X/G]) \simeq \mathrm{QCoh}(BG) \otimes_{\mathsf{Mod}} \mathrm{QCoh}(X)$ by [GR17, Prop. 3.3.5], and ψ^*_X is equivalent to the functor

$$\mathcal{O}_{BG} \boxtimes (-) \colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(BG) \otimes_{\mathsf{Mod}} \mathrm{QCoh}(X)$$

Since $(\varphi^G_X)^* \circ i_{[X/G]}$ is equivalent to the functor $N \boxtimes K \to N \otimes K$ from $\mathrm{QCoh}(BG) \otimes_{\mathsf{Mod}} \mathrm{QCoh}(X)$ to $\mathrm{QCoh}^G(X)$, the claim follows.

(iii). The functor $(-)^{\mathrm{mv}} \colon \mathrm{QCoh}^G(X) \to \mathrm{QCoh}^{\mathrm{mv}}(X)$ preserves colimits, hence has a right adjoint, say $j \colon \mathrm{QCoh}^{\mathrm{mv}}(X) \to \mathrm{QCoh}^G(X)$. For $M \in \mathrm{QCoh}^{\mathrm{mv}}(X)$, take $N \in \mathrm{QCoh}^G(X)$ and an exact sequence $N^{\mathrm{fix}} \to N \to N^{\mathrm{mv}} = jM$. Consider the following commutative diagram

$$\begin{tikzcd}
N^{\mathrm{fix}} \arrow[r] \arrow[d] & N^{\mathrm{fix}} \arrow[r] \arrow[d] & (jM)^{\mathrm{fix}} \arrow[d] \\
N^{\mathrm{fix}} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & jM \arrow[d] \\
0 \arrow[r] & N^{\mathrm{mv}} \arrow[r] & (jM)^{\mathrm{mv}}
\end{tikzcd}$$

The middle row is exact, hence so is the top one by exactness and idempotency of $(-)^{\mathrm{fix}}$, and the right column is exact by definition. It follows that $(jM)^{\mathrm{fix}} = 0$, hence $jM \simeq (jM)^{\mathrm{mv}}$. The diagram also shows that $(-)^{\mathrm{mv}} \circ (-)^{\mathrm{fix}} \simeq 0$.

By construction, for any $K \in \mathrm{QCoh}^G(X)$, it holds that $K^{\mathrm{mv}} \simeq 0$ if and only if $K^{\mathrm{fix}} \to K$ is an equivalence. Since i is fully faithful, the sequence

$$\mathrm{QCoh}(X) \xrightarrow{i} \mathrm{QCoh}^G(X) \xrightarrow{(-)^{\mathrm{mv}}} \mathrm{QCoh}^{\mathrm{mv}}(X)$$

is thus split exact, by Remark 4.9. □

Corollary 4.11. *Let G act trivially on a derived algebraic stack X. Then $(-)^{\mathrm{fix}}$ is also a left adjoint of i, and for each $\mathcal{F} \in \mathrm{QCoh}^G(X)$ the canonical sequence*

$$\mathcal{F}^{\mathrm{fix}} \to \mathcal{F} \to \mathcal{F}^{\mathrm{mv}} \tag{4.4}$$

is split.

Proof. By Remark 4.9, this is immediate from Proposition 4.10. □

Example 4.12. Let X be a derived algebraic stack with a G-action. Then the equivalence $\mathrm{QCoh}^G(X) \simeq \mathrm{QCoh}([X/G])$ identifies $\mathcal{L}_{[X/G]}$ with $\mathcal{L}_X$ endowed with canonical G-action.

4.3. **Relative fixed loci of derived stacks.** Let $f\colon G \to G'$ be a surjective homomorphism of reductive algebraic groups, and $\varphi\colon BG \to BG'$ the induced map on stacks.

Definition 4.13. Let $f_*\colon \mathsf{Stk}^G \to \mathsf{Stk}^{G'}$ be the functor that sends a derived stack $X\colon \mathsf{Alg}^G \to \mathsf{Spc}$ to

$$\mathsf{Alg}^{G'} \xrightarrow{\mathrm{Res}_\varphi} \mathsf{Alg}^G \xrightarrow{X} \mathsf{Spc}$$

Let B, Ω be as in Notation 4.5. Write φ_* for the functor $Bf_*\colon \mathsf{Stk}_{BG} \to \mathsf{Stk}_{BG'}$. Conversely, let $\varphi^*\colon \mathsf{Stk}_{BG'} \to \mathsf{Stk}_{BG}$ be the pullback functor along φ, and let f^* be the functor $\Omega\varphi^*\colon \mathsf{Stk}^{G'} \to \mathsf{Stk}^G$.

Observe that the G-action on $f^*(X)$, for $X \in \mathsf{Stk}^{G'}$, is given by restricting the G'-action on X along $f\colon G \to G'$. It follows that f^* restricts to classical stacks.

Lemma 4.14. *The functor $\mathrm{Res}_\varphi\colon \mathsf{Alg}^{G'} \to \mathsf{Alg}^G$ is fully faithful.*

Proof. Since $\mathrm{Res}_\varphi\colon \mathsf{Mod}^{G'} \to \mathsf{Mod}^G$ is t-exact, we can restrict it to a functor between connective objects, written $H\colon \mathsf{Mod}^{G'}_{\geq 0} \to \mathsf{Mod}^G_{\geq 0}$. It suffices to show that H is fully faithful. Let $h\colon \mathsf{Rep}^{G'} \to \mathsf{Rep}^G$ be the classical restriction functor.

Since H preserves colimits, it is determined by its restriction to $\mathsf{Rep}^{G'}$. Since H restricted to $\mathsf{Rep}^{G'}$ is h, we see that H is the left derived functor of h by [HTT, Prop. 5.5.8.15]. By [HTT, Prop. 5.5.8.22], it therefore suffices to show that $H\colon \mathsf{Rep}^{G'} \to \mathsf{Mod}^G_{\geq 0}$ is fully faithful and with essential image contained in the compact projective objects of $\mathsf{Mod}^G_{\geq 0}$.

We already know that the essential image of H on $\mathsf{Rep}^{G'}$ consists of compact projective objects, since $H = h$ on $\mathsf{Rep}^{G'}$. What remains to show is that h is fully faithful, which follows from the fact that f is surjective. □

Proposition 4.15. *The functors f^*, f_* fit into an adjunction*

$$f^* : \mathsf{Stk}^{G'} \leftrightarrows \mathsf{Stk}^G : f_*$$

where f^* is fully faithful. Consequently, the functor φ_* is the Weil restriction along φ.

Proof. Take a left adjoint F to f_* via left Kan extensions. Since Res_φ is fully faithful by Lemma 4.14, it holds that F and f^* agree when restricted to $\mathsf{Aff}^{G'}$. Since both functors commute with colimits and $\mathsf{Aff}^{G'}$ generates $\mathsf{Stk}^{G'}$ under colimits, F and f^* therefore agree on all of $\mathsf{Stk}^{G'}$. □

4.4. **Absolute fixed loci of derived stacks.** Consider now, for a reductive algebraic group G, the adjunction induced by the canonical map $G \to *$ via Proposition 4.15, written as

$$\iota : \mathsf{Stk} \leftrightarrows \mathsf{Stk}^G : (-)^G$$

Remark 4.16. Let $U \in \mathsf{Stk}^G$ and put $X := [U/G]$. Observe that the derived mapping stack $\underline{\mathrm{Map}}(BG, X)$ exists without any set-theoretic issues. Define $\underline{\mathrm{Map}}^G(*, X)$ via the Cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Map}}^G(*, X) \arrow[r] \arrow[d] & \underline{\mathrm{Map}}(BG, X) \arrow[d, "(X \to BG)_*"] \\ * \arrow[r] & \underline{\mathrm{Map}}(BG, BG). \end{tikzcd}$$

Proposition 4.15 and the formula (3.1) for the Weil restriction imply that $\underline{\mathrm{Map}}^G(*, X)$ is the Weil restriction $\psi_* X$ along $\psi \colon BG \to *$. Moreover, it holds

$$\mathsf{Stk}(T, \psi_* X) \simeq \mathsf{Stk}_{BG}(T \times BG, X) \simeq \mathsf{Stk}^G(T, U) \simeq \mathsf{Stk}(T, U^G)$$

giving another description of U^G in terms of mapping stacks.

Assume now that X is algebraic and locally of finite presentation. Observe that $\psi \colon BG \to *$ is formally proper by [HLP19, Prop. 4.3.4], that $\psi_* \mathcal{O}_{BG} \simeq \psi_* \psi^* \mathbb{C} \simeq \mathbb{C}$ since ψ^* is the fully faithful functor $(-)^{\mathrm{tv}} \colon \mathsf{Mod}_{\mathbb{C}} \to \mathsf{Mod}_{\mathbb{C}}^G$, and that ψ_* is of Tor-amplitude ≥ 0 by the previous point. Therefore, because X is locally of finite presentation and all stacks are assumed to have affine diagonal, [HLP19, Thm. 5.1.1] implies that all mapping stacks in the above diagram are algebraic, hence that U^G is as well.

Write clStk for the category of classical stacks, and clStk^G for the category of classical stacks with G-action. We then have an adjunction

$$\iota_{\mathrm{cl}} : \mathsf{clStk} \rightleftarrows \mathsf{clStk}^G : (-)^{G,\mathrm{cl}}$$

where ι_{cl} endows a stack with trivial G-action, and $(-)^{G,\mathrm{cl}}$ takes the classical fixed points. We thus have a diagram

$$\begin{tikzcd} \mathsf{Stk} \arrow[r, "\iota"] \arrow[d, "(-)_{\mathrm{cl}}"'] & \mathsf{Stk}^G \arrow[r, "(-)^G"] \arrow[d, "(-)_{\mathrm{cl},G}"] & \mathsf{Stk} \arrow[d, "(-)_{\mathrm{cl}}"] \\ \mathsf{clStk} \arrow[r, "\iota_{\mathrm{cl}}"'] & \mathsf{clStk}^G \arrow[r, "(-)^{G,\mathrm{cl}}"'] & \mathsf{clStk} \end{tikzcd} \tag{4.5}$$

where $(-)_{\mathrm{cl},G}$ is taking the underlying classical G-stack. We will often write $(-)_{\mathrm{cl},G}$ resp. $(-)^{G,\mathrm{cl}}$ simply as $(-)_{\mathrm{cl}}$ resp. as $(-)^G$, which is justified by the first point of the following result.

Proposition 4.17.

(i) *The diagram (4.5) is commutative.*

(ii) *Suppose that $X \in \mathsf{Stk}^G$ is a derived scheme such that X_{cl} is separated. Then X^G is a derived scheme, and the unit $\iota X^G \to X$ of the adjunction $\iota \dashv (-)^G$ is a closed immersion. In particular, for $B \in \mathsf{Alg}^G$, it holds that $(\mathrm{Spec}^G B)^G$ is affine.*

Proof. (i) Clearly, it holds $\iota_{\mathrm{cl}} \circ (-)_{\mathrm{cl}} \simeq (-)_{\mathrm{cl},G} \circ \iota$. Let now $j\colon \mathrm{cl}\mathsf{Stk} \to \mathsf{Stk}$ and $j^G\colon \mathrm{cl}\mathsf{Stk}^G \to \mathsf{Stk}^G$ be the inclusions. Then also $\iota \circ j = j^G \circ \iota_{\mathrm{cl}}$. By composing adjunctions, we have

$$\iota \circ j \dashv (-)_{\mathrm{cl}} \circ (-)^G \qquad \& \qquad j^G \circ \iota_{\mathrm{cl}} \dashv (-)^{G,\mathrm{cl}} \circ (-)_{\mathrm{cl},G}$$

and therefore $(-)_{\mathrm{cl}} \circ (-)^G \simeq (-)^{G,\mathrm{cl}} \circ (-)_{\mathrm{cl},G}$.

(ii) Let $X \in \mathsf{Stk}^G$ be a derived scheme such that X_{cl} is separated. By [CGP15, Prop. A.8.10] it holds that X_{cl}^G is a closed subscheme of X_{cl}, hence that X^G is a closed subscheme of X. □

We can now give an explicit description of $\mathcal{O}_{(\operatorname{Spec} B)^G}$ for $B \in \mathsf{Poly}^G$. By Proposition 4.17, the adjunction $\iota \dashv (-)^G$ restricts to affine objects. On the algebra side, this gives us an adjunction

$$(-)_G : \mathsf{Alg}^G \leftrightarrows \mathsf{Alg} : (-)^{\mathrm{tv}}$$

where $(-)^{\mathrm{tv}}$ is the functor which endows an algebra with trivial G-action.

Definition 4.18. Let $A \in \mathsf{Alg}^G$ and $(M \in \mathsf{Mod}_A^G)_{\geq 0}$. Define

$$A[M] := \mathrm{LSym}_A^G(M)$$

Let now $B \in \mathsf{Alg}_A^G$ and $\sigma\colon M \to B$ a map of (G,A)-modules be given. Define $B/(\sigma)$ as the pushout

$$\begin{tikzcd} A[M] \arrow[r, "p"] \arrow[d, "\sigma"] & A \arrow[d] \\ B \arrow[r] & B/(\sigma) \end{tikzcd}$$

where p is induced by the map $M \to 0$.

Suppose that $M \in \mathsf{Rep}^G$, say $\dim M = d$. Then $\sigma\colon \mathbb{C}[M[k]] \to B$ induces elements $v_1, \ldots, v_d \in \pi_k(B)$, and the underlying algebra of $B/(\sigma)$ is equivalent to the derived quotient $B/(v_1, \ldots, v_d)$. This follows because forgetting the action is a left adjoint, see Notation 4.7.

Let $B \in \mathsf{Alg}^G$. The underlying module of B is a $(G, \mathbb{C}^{\mathrm{tv}})$-module, hence the sequence $B^{\mathrm{fix}} \to B \to B^{\mathrm{mv}}$ in Mod^G is split by Corollary 4.11.

Proposition 4.19. *For $M \in \mathsf{Mod}_{\geq 0}^G$ it holds $\mathbb{C}[M]_G \simeq \mathbb{C}[M^{\mathrm{fix}}]$.*

Proof. Let $R \in \mathsf{Alg}$. By Proposition 4.10, we have that $(-)^{\mathrm{fix}}$ is a left adjoint of $(-)^{\mathrm{tv}}$. We thus have

$$\mathsf{Alg}(\mathbb{C}[M]_G, R) \simeq \mathsf{Alg}^G(\mathbb{C}[M], R^{\mathrm{tv}}) \simeq \mathsf{Mod}^G(M, R^{\mathrm{tv}}) \simeq \mathsf{Mod}(M^{\mathrm{fix}}, R)$$

from which the claim follows. □

The same methods also imply the following statement about relative fixed loci.

Proposition 4.20. *Let $G \to G'$ be a surjective homomorphism of reductive groups with kernel H and $X \in \mathsf{Aff}^G$. Then H is reductive and the Weil restriction of X along the morphism $BG \to BG'$ is given by the H-fixed locus $[X^H/G']$.*

4.5. **The cotangent complex of the fixed locus.**

Proposition 4.21. *Let $B \in \mathsf{Alg}^G$ be given. Then it holds*

$$\mathbb{L}_{B^G/B} \simeq \operatorname{colim}_{B_\alpha \to B} \big((B_\alpha)_G[1] \otimes_{\mathbb{C}} (M_\alpha)^{\mathrm{mv}}) \otimes_{(B_\alpha)_G} B_G \big)$$

where the indexing category runs over all G-equivariant maps $B_\alpha \to B$, where B_α is of the form $\mathbb{C}[M_\alpha]$ for some $M_\alpha \in \mathsf{Rep}^G$.

Proof. First suppose that $B = \mathrm{Sym}^G(M)$ for some $M \in \mathsf{Rep}^G$, say of dimension n, containing t trivial representations. We know that the map $B \to B_G$ is the map

$$\mathbb{C}[M] \to \mathbb{C}[M^{\mathrm{fix}}]$$

which sends M^{mv} to 0. Since this is quasi-smooth, the claim follows.

The general claim follows from the fact that $\mathbb{L}_{B^G/(-)}$ commutes with colimits, together with Lemma 4.2. $\square$

Proposition 4.22. *Let X be a derived algebraic G-stack, write $i\colon X^G \to X$ for the canonical map. Then the exact sequence*

$$i^*\mathbb{L}_X \to \mathbb{L}_{X^G} \to \mathbb{L}_{X^G/X}$$

is equivalent to the exact sequence

$$i^*\mathbb{L}_X \to (i^*\mathbb{L}_X)^{\mathrm{fix}} \to (i^*\mathbb{L}_X)^{\mathrm{mv}}[1]$$

induced by the splitting $i^*\mathbb{L}_X \simeq (i^*\mathbb{L}_X)^{\mathrm{fix}} \oplus (i^*\mathbb{L}_X)^{\mathrm{mv}}$ from Corollary 4.11.

Proof. By construction, the G-action on X^G is trivial. Let $j\colon [X^G/G] \to [X/G]$ and $p\colon [X^G/G] \to X^G$ be the canonical maps. By Proposition 4.10, the functor j^* is equivalent to the G-equivariant pullback $(j^G)^*$, and the functor p_* is equivalent to $(-)^{\mathrm{fix}}$. It follows that $p_*j^*\mathbb{L}_{[X/G]/BG} \simeq (i^*\mathbb{L}_X)^{\mathrm{fix}}$.

Recall from Remark 4.16 that the Weil restriction $\psi_*[X/G]$ along $\psi\colon BG \to *$ is X^G. Then the pullback of ψ is the map p, and j is induced by the counit of the adjunction $\psi^* \dashv \psi_*$. Moreover, by Proposition 4.10, it holds that p_* is also a left adjoint of p^*. Therefore, by the formula for the cotangent complex of Weil restrictions given in [HKR22], it also holds $p_*j^*\mathbb{L}_{[X/G]/BG} \simeq \mathbb{L}_{X^G}$. The claim follows. $\square$

4.6. **Base-change of fixed loci.** We have the following étale base change lemma. Let G be a reductive algebraic group as usual.

Lemma 4.23. *Let $U \to V$ be a G-equivariant morphism of derived schemes, write $f\colon [U/G] \to [V/G]$ for the induced map, and consider the natural commutative square*

$$\begin{tikzcd} {[U^{G^0}/G]} \arrow[r] \arrow[d] & {[V^{G^0}/G]} \arrow[d] \\ {[U/G]} \arrow[r, "f"] & {[V/G]}, \end{tikzcd}$$

where $G^0 \subseteq G$ denotes the identity component of G. If f is étale and f_{cl} is separated, then the square is Cartesian.

Proof. Write $X = [U/G]$ and $Y = [V/G]$. By naturality of $(-)^{G^0}$, if $V^{G^0} = \emptyset$, then $U^{G^0} = \emptyset$, and there is nothing to prove. So we assume that $V^{G^0} \neq \emptyset$. Now $U^{G^0} \neq \emptyset$ if and only if X and Y have the same maximal stabilizer dimension. Hence, this is true for the underlying classical truncations, since we have a natural isomorphism $[U_{\mathrm{cl}}^{G^0}/G] = (X_{\mathrm{cl}})^{\max} \to (Y_{\mathrm{cl}})^{\max} \times_{Y_{\mathrm{cl}}} X_{\mathrm{cl}} = [V_{\mathrm{cl}}^{G^0}/G] \times_{Y_{\mathrm{cl}}} X_{\mathrm{cl}}$ by the discussion following the statement of Proposition C.5 in [ER21].

By G-equivariance and étaleness of the morphism $X \to Y$, we have a commutative diagram

$$\begin{array}{ccccc}
\mathbb{L}_Y|_{X^{G^0}} & \simeq & (\mathbb{L}_Y|_{Y^{G^0}})^{\mathrm{fix}}|_{X^{G^0}} & \oplus & (\mathbb{L}_Y|_{Y^{G^0}})^{\mathrm{mv}}|_{X^{G^0}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}_X|_{X^{G^0}} & \simeq & (\mathbb{L}_X|_{X^{G^0}})^{\mathrm{fix}} & \oplus & (\mathbb{L}_X|_{X^{G^0}})^{\mathrm{mv}}
\end{array}$$

where the vertical arrows are equivalences and the splittings are with respect to the G^0-action.

But $\mathbb{L}_{X^{G^0}} \simeq (\mathbb{L}_X|_{X^{G^0}})^{\mathrm{mv}}$ and $\mathbb{L}_{Y^{G^0}} \simeq (\mathbb{L}_Y|_{Y^{G^0}})^{\mathrm{mv}}$ and it thus follows that $\mathbb{L}_{X^{G^0}/Y^{G^0}} \simeq 0$ and $X^{G^0} \to Y^{G^0}$ is étale.

Now $X^{G^0} \to Y^{G^0}$ factors through the étale morphism $X \times_Y Y^{G^0} \to Y^{G^0}$, so the map $X^{G^0} \to X \times_Y Y^{G^0}$ must also be étale and hence an isomorphism, as it induces an isomorphism on classical truncations. $\square$

4.7. **Equivariant standard forms.** Our discussion so far allows us to give natural equivariant extensions of standard forms and their existence.

To this end, let mod^G be the 1-category of presheaves on Rep^G that send finite coproducts in Rep^G to products of sets. Then mod^G is the category of discrete G-modules $(\mathsf{Mod}^G)^\heartsuit$, and Mod^G is the unbounded derived category of mod^G. A *G-cdga* is a commutative algebra object in the category $\mathrm{Ch}_{\geq 0}(\mathsf{mod}^G)$ of chain complexes in homological degree ≥ 0, with the natural symmetric monoidal structure. Similarly, we define a *graded-commutative G-ring* as a graded-commutative algebra object in mod^G.

Definition 4.24. Let an algebra $A \in \mathsf{Alg}^G$ and $x \in \operatorname{Spec} A$ fixed by G be given. A G-cdga model R of A is said to be *in standard form* if R_0 is smooth with Ω_{R_0} free, and the underlying graded-commutative G-ring of R is freely generated over R_0 on a finite number of generators. If this holds, then we say that R is *minimal at x* if the underlying cdga (which is non-equivariantly in standard form) is minimal at x.

Remark 2.5 also holds in the equivariant case. That is, if a G-cdga R is in standard form, then it can be obtained from R_0 by equivariant cell-attachements $R(0) \to R(1) \to \cdots \to R(n)$ such that, for all k, the underlying graded-commutative G-ring of $R(k)$ is the subring of the underlying graded-commutative G-ring of R generated by elements in homogeneous degree $\leq k$, and M_{k+1} is a free G-module over $R(k)$.

As in the non-equivariant case, we say that $B \in \mathsf{Alg}_A^G$ (or the corresponding map on spectra) is *locally of finite G-presentation* if B is a compact object in Alg_A^G. Likewise, we call B *finitely G-presented* if it is contained in the smallest subcategory of Alg_A^G that contains Poly_A^G and is closed under finite colimits. We note that any classical, finitely presented, affine scheme $T \in \mathsf{Stk}^G$ admits a G-equivariant closed embedding into a smooth classical scheme $\operatorname{Spec}^G(\mathbb{C}[M])$ for some $M \in \mathsf{Rep}^G$. Moreover, B is finitely G-presented if and only if B is finitely presented and admits a G-action (one can see this by the argument sketched in the proof of the following lemma). Then $\operatorname{Spec}^G B \to \operatorname{Spec}^G A$ is locally of finite G-presentation if and only if $\operatorname{Spec}^G B$ is Zariski locally in Stk^G finitely G-presented over $\operatorname{Spec}^G A$. We then have the following.

Lemma 4.25. *Let an algebra $A \in \mathsf{Alg}^G$ and $x \in \operatorname{Spec} A$ fixed by G be given. Suppose that A is locally of finite presentation over $\mathbb{C}$. Then, up to equivariant Zariski localizing around x, there is a G-cdga model R of A in standard form. We can arrange R to be such that $R_0 = \mathbb{C}[M]$ for some $M \in \mathsf{Rep}^G$. Alternatively, we can arrange R to be minimal at x.*

In addition, in both cases, up to further equivariant Zariski shrinking, the underlying $R_{\leq i}$-modules M_{i+1} may be taken, for all $i \geq 0$, in the form $M_{i+1} = R_{\leq i} \otimes_{\mathbb{C}} W_{i+1}$ for G-representations $W_{i+1} \in \mathsf{Rep}^G$.

Proof. The same argument that proves Lemma 2.7 applies equivariantly to establish everything, but the last assertion.

We briefly sketch how to equivariantly adapt the steps of the proof of [BBJ19, Theorem 4.1]. Write $X = \operatorname{Spec}^G A$ for brevity. Since A is locally of finite G-presentation over $\mathbb{C}$, we may assume, up to G-equivariant Zariski open localizing around x, that A is finitely G-presented. Following [BBJ19, Theorem 4.1], we may now either choose a G-equivariant closed embedding $X_{\mathrm{cl}} = \pi_0 X \to \operatorname{Spec}^G(\mathbb{C}[M])$ for some $M \in \mathsf{Rep}^G$ or, using the reductivity of G, after further equivariant Zariski open localization around $x \in X_{\mathrm{cl}}$, a G-equivariant closed embedding $X_{\mathrm{cl}} = \pi_0(X) \to \operatorname{Spec}^G(R_0)$ into a smooth scheme, that is minimal at x for some finitely presented $R_0 \in \mathsf{Alg}^G$.

We now continue with the minimal case, as the case with $R_0 = \mathbb{C}[M]$ proceeds similarly. We observe that since $[\operatorname{Spec}^G(R_0)/G]$ is smooth over BG, the closed embedding $[X_{\mathrm{cl}}/G] \to [\operatorname{Spec}^G(R_0)/G]$ can be lifted over BG to a closed embedding $[X/G] \to [\operatorname{Spec}^G(R_0)/G]$, which by Proposition 4.4 amounts to a G-equivariant map $R_0 \to A$, lifting $R_0 \to \pi_0(A)$. Let F be the fiber of $R_0 \to A$. Then, up to possible further equivariant Zariski open shrinking around x, there exists a free G-module M_1 over R_0 that surjects onto $\pi_0(F)$ and whose fiber at x is isomorphic to $\pi_0(F)|_x$. For this to be possible, we use the reductivity of G and that x is fixed by G. The morphism $M_1 \to \pi_0(F) \to R_0$ determines a ring $R(1)$, together with a G-equivariant morphism $R(1) \to A$ that is an isomorphism on π_0. This showcases the first step of the construction. One argues similarly to construct the remaining $R(i)$, by following the proof of [BBJ19, Theorem 4.1].

To establish the last assertion, it clearly suffices to prove that for any G-equivariant vector bundle $\mathcal{W}$ on a derived affine G-scheme U there exists a G-invariant Zariski open neighbourhood $x \in U_x \subseteq U$ and a G-equivariant isomorphism $\mathcal{W}|_{U_x} \cong W \otimes_{\mathbb{C}} \mathcal{O}_U$, where W is a G-representation.

Since x is fixed by G, G acts on the fiber $W := \mathcal{W}|_x$ and we have a G-equivariant surjection $\mathcal{W} = \Gamma(U, \mathcal{W}) \to W$. Since G is reductive, we may split this surjection and consider $W \subseteq \Gamma(U, \mathcal{W})$. This induces a G-equivariant morphism $W \otimes_{\mathbb{C}} \mathcal{O}_U \to \mathcal{W}$. For U classical, the locus where this map is an isomorphism is clearly a G-invariant open subscheme of U which contains x, and the general case reduces to the classical case, since $\mathcal{W}$ is a vector bundle. This concludes the proof. $\square$

Now, our prior discussion of fixed loci immediately implies an explicit description for $(\operatorname{Spec} B)^G$ when B has a model in standard form, given in the next propositions.

For R a model in standard form for $B \in \mathsf{Alg}$, we write $B(i)$ for the algebra in Alg corresponding to the cdga $R(i)$.

Proposition 4.26. *Suppose that $B \in \mathsf{Alg}^G$ has a model R in standard form such that $M_{i+1} = R(i) \otimes_{\mathbb{C}} W_{i+1}$ for some G-representations W_{i+1}. Then B_G also has a model R_G in standard form with $(R_G)_0 = \mathcal{O}_{(\operatorname{Spec} R_0)^G}$ and generated by the modules N_{i+1} defined inductively by $N_{i+1} = R(i)_G \otimes_{\mathbb{C}} W_{i+1}^{\mathrm{fix}}$, where $R(i)_G$ is the model in standard form for $B(i)_G$ generated by N_j for $j \leq i$.*

In addition, for a point $x \in \operatorname{Spec} B$ fixed by G, if R is minimal at x, we can arrange R_G to be minimal at x as well.

Proof. Recall that R_0 is smooth and classical by definition. We first show that $(\operatorname{Spec} R_0)^G$ is smooth and classical and coincides with the classical fixed locus. To this end, write $X = \operatorname{Spec} R_0$. Then, by [Dré04, Lemma 5.1], for any G-fixed point x, there is a Zariski open G-invariant neighbourhood $U_x \subseteq X$ together with an étale G-equivariant morphism $U_x \to T_x X$, where $T_x X$ denotes the tangent space of X at x. Let U be the disjoint union of the U_x as x ranges over the fixed points of X. We obtain an open G-equivariant immersion $f\colon U \to X$. Since $f = f_{\mathrm{cl}}$ is separated, by Lemma 4.23, $U^G = X^G \times_X U$. Applying the same lemma to each morphism $U_x \to T_x X$, which is étale and separated, we get $(U_x)^G = U_x \times_X (T_x X)^G$. By Proposition 4.19, since $T_x X = \operatorname{Spec}(\mathbb{C}[M])$ for some $M \in \mathsf{Rep}^G$, it follows that $(T_x X)^G$ is smooth and classical, and hence so is $(U_x)^G$, and therefore U^G as well. Then, $U^G = X^G \times_X U$, and the fact that $U^G \to X^G$ is surjective on points by construction, implies that X^G must be smooth and classical. In particular, it must coincide with the classical fixed locus in this case.

For $i \geq 0$, we assume by induction that the cdga $R(i)_G$, generated by N_j for $j \leq i$, is a model for $B(i)_G$, and put $N_{i+1} \coloneqq R(i)_G \otimes_{\mathbb{C}} W_{i+1}^{\mathrm{fix}}$. Then we have a (strictly) commutative diagram of cdgas

$$
\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d] & \mathbb{C}[W_{i+1}[i]] \arrow[r, "W_{i+1}[i]\mapsto 0"] \arrow[d, "\alpha"] & \mathbb{C} \arrow[d] \\
R(i) \arrow[r] & \operatorname{Sym}_{R(i)}(M_{i+1}[i]) \arrow[r, "M_{i+1}[i]\mapsto 0"] \arrow[d] & R(i) \arrow[d] \\
 & R(i) \arrow[r] & R(i+1)
\end{tikzcd}
\tag{4.6}
$$

where the square on the left is the natural one coming from the fact that $M_{i+1} = R(i) \otimes W_{i+1}$, and the bottom square comes from the definition of

$R(i+1)$. One shows that all squares are homotopy pushouts, using that M_{i+1} is free and [BBJ19, §2.2]. Define $R(i+1)_G$ as the homotopy pushout of the map $\mathbb{C}[W^{\mathrm{fix}}_{i+1}[i]] \to R(i)_G$ (induced by α) along the map $\mathbb{C}[W^{\mathrm{fix}}_{i+1}[i]] \to \mathbb{C}$ induced by the zero map. Then, since $(-)_G$ commutes with pushouts, we see that $R(i+1)_G$ is a model for $B(i+1)_G$, generated by N_j for $j \leq i+1$. This concludes the first claim.

Let a point $x \in \operatorname{Spec} B$ be fixed by G such that R is minimal at x. By [BBJ19, Prop. 2.12], the cotangent complex for R restricted to x has underlying chain complex

$$\cdots \to W_{i+1}|_x \to W_i|_x \to W_{i-1}|_x \to \cdots$$

and the cotangent complex for R_G restricted to x has underlying chain complex

$$\cdots \to W^{\mathrm{fix}}_{i+1}|_x \to W^{\mathrm{fix}}_i|_x \to W^{\mathrm{fix}}_{i-1}|_x \to \cdots$$

The claim thus holds by definition of minimality. □

4.8. **Equivariant blow-ups.** As mentioned in §4.1, the ∞-category Mod^G is a derived algebraic context, as in [Rak20]. We can thus apply the machinery internally to Mod^G to get G-equivariant Rees algebras and blow-ups, as done in [BH22] in a more general setting. Let us formulate this in our setting, and compare with the non-equivariant case.

Let $X \in \mathsf{Stk}^G$ be given. Write ζ^G_X for the G-equivariant map $X \times [\mathbb{A}^1/\mathbb{G}_m] \to X \times B\mathbb{G}_m$ induced by the zero section. We then have an adjunction

$$(\zeta^G_X)^* : \mathsf{Stk}^G_{X\times[\mathbb{A}^1/\mathbb{G}_m]} \leftrightarrows \mathsf{Stk}^G_{X\times B\mathbb{G}_m} : \mathrm{Res}_{\zeta^G_X}$$

where $(\zeta^G_X)^*$ is the pullback functor, $\mathrm{Res}_{\zeta^G_X}$ is the G-equivariant Weil restriction functor, and we endow $[\mathbb{A}^1/\mathbb{G}_m]$ and $B\mathbb{G}_m$ with trivial G-action.

Let $Y \in \mathsf{Stk}^G$ be given, with trivial G-action. Consider the adjunction $[-/G] : \mathsf{Stk}^G \leftrightarrows \mathsf{Stk} : \iota$, where ι endows a derived stack with trivial G-action. Taking slices, we have an adjunction

$$[-/G] : \mathsf{Stk}^G_{X\times Y} \leftrightarrows \mathsf{Stk}_{[X/G]\times Y} : \iota$$

by [HTT, Prop. 5.2.4.4].

Lemma 4.27. *Let $Z \to X$ in Stk^G be given. Then it holds*

$$[\mathrm{Res}_{\zeta^G_X}(Z \times B\mathbb{G}_m)/G] \simeq \mathrm{Res}_{\zeta_X}[(Z \times B\mathbb{G}_m)/G]$$

Proof. Let T be a stack over $[X/G]\times[\mathbb{A}^1/\mathbb{G}_m]$, and write $P \to X \times [\mathbb{A}^1/\mathbb{G}_m]$ for the associated G-torsor. Then it holds

$$\begin{aligned}
&\mathsf{Stk}_{[X/G]\times[\mathbb{A}^1/\mathbb{G}_m]}(T, [\mathrm{Res}_{\zeta^G_X}(Z \times B\mathbb{G}_m)/G]) \\
&\quad\simeq \mathsf{Stk}^G_{X\times[\mathbb{A}^1/\mathbb{G}_m]}(P, \mathrm{Res}_{\zeta^G_X}(Z \times B\mathbb{G}_m)) \\
&\quad\simeq \mathsf{Stk}^G_{X\times B\mathbb{G}_m}(P \times_{[\mathbb{A}^1/\mathbb{G}_m]} B\mathbb{G}_m, Z \times B\mathbb{G}_m) \\
&\quad\simeq \mathsf{Stk}_{[X/G]\times B\mathbb{G}_m}(T \times_{[\mathbb{A}^1/\mathbb{G}_m]} B\mathbb{G}_m, [Z \times B\mathbb{G}_m/G]) \\
&\quad\simeq \mathsf{Stk}_{[X/G]\times[\mathbb{A}^1/\mathbb{G}_m]}(T, \mathrm{Res}_{\zeta_X}[(Z \times B\mathbb{G}_m)/G])
\end{aligned}$$

from which the claim follows. □

For $\mathcal{B} \in (\mathrm{QAlg}^G)^{\mathbb{N}}(X)$, the *$G$-equivariant projective spectrum* is the stack quotient

$$\mathrm{Proj}^G(\mathcal{B}) \coloneqq [((\mathrm{Spec}^G \mathcal{B}) \setminus V(\mathcal{B}_+))/\mathbb{G}_m]$$

taken inside Stk^G, where $\mathbb{G}_m$ is endowed with trivial G-action, and $V(\mathcal{B}_+) = \mathrm{Spec}^G(\mathcal{B}_0)$ by definition.

Let $D^G_{Z/X} \to X \times \mathbb{A}^1$ be the $\mathbb{G}_m$-torsor (in Stk^G) associated to $\mathrm{Res}_{\zeta^G_X}(Z \times B\mathbb{G}_m)$. Then $D^G_{Z/X} \to X \times \mathbb{A}^1$ is affine, as shown in [BH22]. Let, then, the *G-equivariant extended Rees algebra* be the quasi-coherent $(G, \mathcal{O}_X[t^{-1}])$-algebra $\mathcal{R}^{G,\mathrm{ext}}_{Z/X}$ such that $\mathrm{Spec}^G(\mathcal{R}^{G,\mathrm{ext}}_{Z/X}) \simeq D_{Z/X}$, and put $\mathcal{R}^G_{Z/X} \coloneqq (\mathcal{R}^{G,\mathrm{ext}}_{Z/X})_{\geq 0}$. Then we define the *$G$-equivariant blow-up of X in Z* as the G-equivariant projective spectrum

$$\mathrm{Bl}^G_{Z/X} \coloneqq \mathrm{Proj}^G(\mathcal{R}^G_{Z/X})$$

As in the non-equivariant case, we write $R^{G,\mathrm{ext}}_{B/A}$ etc. in the affine case.

Corollary 4.28. *It holds that* $[\mathrm{Bl}^G_{Z/X}/G] \simeq \mathrm{Bl}_{[Z/G]/[X/G]}$.

Proof. Lemma 4.27 says that

$$\mathrm{Spec}(\mathcal{R}_{[Z/G]/[X/G]}) \simeq [\mathrm{Spec}^G(\mathcal{R}^G_{Z/X})/G]$$

from which the claim follows. □

Proposition 4.29. *Take $A \in \mathsf{Alg}^G$ and let $\sigma\colon M \to A$ be a map of connective (G, A)-modules. Put $B \coloneqq A/(\sigma)$. Then we can compute the G-equivariant extended Rees algebra of B over A as*

$$R^{G,\mathrm{ext}}_{B/A} \simeq \frac{A[t^{-1}, M]}{(t^{-1}M - \sigma)}$$

where the quotient is with respect to the map $M \to A[t^{-1}, M]$ which is induced by substracting $\sigma\colon M \to A \to A[t^{-1}, M]$ from the map

$$\mathrm{LSym}^G_{A[t^{-1}]}(\times t^{-1})\colon \mathrm{LSym}^G_{A[t^{-1}]}(M[t^{-1}]) \to \mathrm{LSym}^G_{A[t^{-1}]}(M[t^{-1}])$$

One can show this along similar lines as done in [Hek21] for the formula, presented in Example 3.6, of the extended Rees algebra of a finite quotient in the non-equivariant case. However, the theory of blow-ups in nonconnective derived algebraic geometry from [BH22]—which is based on [Rak20]—gives a more straightforward argument. Let us review the necessary ingredients to carry this out.

Write DAlg^G for the ∞-category of (nonconnective) derived rings in Mod^G. By definition, LSym^G extends to a monadic adjunction

$$\mathrm{LSym}^G : \mathsf{Mod}^G \leftrightarrows \mathsf{DAlg}^G : U$$

where U is the forgetful functor. Since we are in characteristic zero, DAlg^G is equivalent to the ∞-category of $\mathbb{E}_\infty$-algebras in Mod^G. We use similar notation as for the connective settings, writing DAlg^G_A for the slice category $\mathsf{DAlg}^G_{A/}$ etc.

Let $A \in \mathsf{Alg}^G$ be given. The key ingredient we will be using is an extension of the extended Rees algebra to a colimit preserving functor

$$R^{G,\mathrm{ext}}_{(-/A)} \colon \mathsf{Alg}^G_A \to (\mathsf{DAlg}^G_{A[t^{-1}]})^{\mathbb{N}}$$

which has the property that

$$R^{G,\mathrm{ext}}_{\mathrm{LSym}^G_A(M)/A} \simeq \mathrm{LSym}^G_{A[t^{-1}]}(M[-1])$$

for any connective (G, A)-module M.

Proof of Proposition 4.29. By definition, $B = A/(\sigma)$ fits into the following pushout diagram in Alg^G

$$\begin{tikzcd} A[M] \arrow[r, "z"] \arrow[d, "s"] & A \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

where z is the map induced by $0 \colon M \to A$, and s is induced by $\sigma \colon M \to A$. It follows that $R^{G,\mathrm{ext}}_{B/A}$ fits into the following pushout diagram in $(\mathsf{DAlg}^G_{A[t^{-1}]})^{\mathbb{N}}$

$$\begin{tikzcd} \mathrm{LSym}^G_{A[t^{-1}]}(M[-1]) \arrow[r, "\varphi"] \arrow[d, "\psi"] & A[t^{-1}] \arrow[d] \\ A[t^{-1}] \arrow[r] & R^{G,\mathrm{ext}}_{B/A} \end{tikzcd} \tag{4.7}$$

where ψ is induced by the zero map, and φ is induced by the diagram

$$\begin{tikzcd} M[t^{-1}][-1] \arrow[r] \arrow[d, "\times t^{-1}[-1]"] & 0 \arrow[d] & \\ M[t^{-1}][-1] \arrow[r] \arrow[d] & M[-1] \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[d] \\ 0 \arrow[r] & M[t^{-1}] \arrow[r, "\times t^{-1}"] \arrow[dr, "{\sigma[t^{-1}]}"'] & M[t^{-1}] \\ & & A[t^{-1}] \end{tikzcd}$$

consisting of pushouts, where the map $\sigma[t^{-1}]$ is induced by $\sigma \colon M \to A$, and φ corresponds to the composition $\sigma[t^{-1}] \circ \alpha$ under the adjunction $\mathrm{LSym}^G_{A[t^{-1}]} \dashv U$, where U is the forgetful functor. We can thus rewrite the diagram in (4.7) as

$$\begin{tikzcd} \mathrm{LSym}^G_{A[t^{-1}]}(M[-1]) \arrow[r, "\alpha'"] \arrow[d, "w"] & \mathrm{LSym}^G_{A[t^{-1}]}(M[t^{-1}]) \arrow[r, "\sigma'"] \arrow[d, "\gamma"] & A[t^{-1}] \arrow[d] \\ A[t^{-1}] \arrow[r] & \mathrm{LSym}^G_{A[t^{-1}]}(M[t^{-1}]) \arrow[r] & R^{G,\mathrm{ext}}_{B/A} \end{tikzcd}$$

where w is induced by $0\colon M[-1] \to A[t^{-1}]$, α' is induced by α, and σ' is induced by $\sigma[t^{-1}]$, and both squares are pushouts.

By construction, γ is then obtained by applying $\mathrm{LSym}^G_{A[t^{-1}]}$ to the map $\times t^{-1}\colon M[t^{-1}] \to M[t^{-1}]$. By the universal property of $\mathrm{LSym}^G_{A[t^{-1}]}$, this corresponds to a map

$$t^{-1}M\colon M[t^{-1}] \to \mathrm{LSym}^G_{A[t^{-1}]}[M[t^{-1}]] \simeq A[t^{-1}, M]$$

Write σ for the map $M[t^{-1}] \to A[t^{-1}, M]$ induced by $\sigma[t^{-1}]$. Then, rewriting once more, we indeed have a pushout diagram

$$\begin{tikzcd} A[t^{-1}, M] \arrow[r, "v"] \arrow[d, "\tau"] & A[t^{-1}] \arrow[d] \\ A[t^{-1}, M] \arrow[r] & R^{G,\mathrm{ext}}_{B/A} \end{tikzcd}$$

where v is induced by $0\colon M \to A[t^{-1}]$ and τ by $t^{-1}M - \sigma\colon M[t^{-1}] \to A[t^{-1}, M]$. $\square$

Example 4.30. Suppose that $M \in (\mathsf{Mod}^G_A)_{\geq 0}$ is of the form $M = M_1[k_1] \oplus \cdots \oplus M_n[k_n]$, where each $M_i \in \mathsf{Rep}^G_A$ is of the form $M_i = A \otimes V_i$, for $V_i \in \mathsf{Rep}^G$ of dimension d_i, and $k_1, \ldots, k_n \in \mathbb{N}$. For each i, let $\underline{v}_i = (v_{i1}, \ldots, v_{id_i})$ be a basis for the underlying A-module of M_i.

Let a map $\sigma\colon M \to A$ be given, and let B be the quotient $A/(\sigma)$. Then the restrictions $M_i[k_i] \to A$ of σ induce elements $\underline{a}_i = (a_{i1}, \ldots, a_{id_i}) \in \pi_{k_i}(A)$. From Proposition 4.29, it follows that the underlying $A[t^{-1}]$-algebra of $R^{G,\mathrm{ext}}_{B/A}$ is

$$\frac{A[t^{-1}, \underline{x}_1, \ldots, \underline{x}_n]}{(\underline{x}_1 t^{-1} - \underline{a}_1, \ldots, \underline{x}_n t^{-1} - \underline{a}_n)}$$

where each $\underline{x}_i$ is a sequence of free variables $x_{i1}, \ldots, x_{id_i}$ in homogeneous degree 1 and homological degree k_i.

Example 4.31. Let $A \in \mathsf{Alg}^G$ be given. Take a connective (G, A)-module M of the form $M = M_1[k_1] \oplus \cdots \oplus M_n[k_n]$, with each $M_i \in \mathsf{Rep}^G_A$, say $M_i = A \otimes V_i$ for some $V_i \in \mathsf{Rep}^G$. Let $\sigma\colon M \to A$ be a given map of (G, A)-modules, and put $B \coloneqq A/(\sigma)$. We will give a formula for $R^{G,\mathrm{ext}}_{B_G/B}$.

Let $\tau\colon B[B^{\mathrm{mv}}] \to B$ be the induced map of (G, B)-algebras. Put $V \coloneqq \bigoplus_{1 \leq i \leq n} V_i$. Observe that

$$A[M]_G \simeq (\mathrm{Sym}^G_A(A \otimes V))_G \simeq (\mathrm{Sym}^G(V) \otimes A)_G \simeq A_G[M^{\mathrm{mv}}]$$

It follows that B_G fits into the following pushout square

$$\begin{tikzcd} A_G[M^{\mathrm{mv}}] \arrow[r, "s"] \arrow[d, "z"] & A_G \arrow[d] \\ A_G \arrow[r] & B_G \end{tikzcd}$$

where z is induced by the zero map and s by the restriction of σ to M^{mv}. It follows B^{mv} is the cofibre of $M^{\mathrm{mv}} \to A^{\mathrm{mv}}$.

Write M as $M' \oplus M''$, where M' is the direct sum of those $M_i[k_i] \subset M$ for which the restriction $M' \to A$ is nonzero. Then $M'' \to A$ is zero. It follows that

$$B^{\mathrm{mv}} \simeq N := A^{\mathrm{mv}}/M'' \oplus M'[1]$$

hence, by Example 4.30,

$$R^{G,\mathrm{ext}}_{B_G/B} \simeq \frac{B[t^{-1}, N]}{(t^{-1}N - \tau)}.$$

4.9. **A midway vista.** We have a diagram with commutative solid part

$$\begin{tikzcd}
\mathsf{Stk} \arrow[r, "\iota"] & \mathsf{Stk}^G \arrow[l, bend right, "{[-/G]}"'] \arrow[l, bend left, "(-)^G"] \arrow[r, "U"] & \mathsf{Stk} \arrow[l, bend right, "(-)\times G"'] \arrow[l, bend left, "\underline{\mathrm{Map}}(G,-)"] \\
\mathsf{Aff} \arrow[u] \arrow[r, "\iota"] & \mathsf{Aff}^G \arrow[u] \arrow[l, bend right, dashed, "\mathrm{Spec}\,\mathrm{cInd}_\psi"'] \arrow[l, bend left, "(-)^G"] \arrow[r, "U"] & \mathsf{Aff} \arrow[u] \arrow[l, bend right, "(-)\times G"']
\end{tikzcd}$$

with adjunctions $[-/G] \dashv \iota \dashv (-)^G$, and likewise for the other parts of the diagram, and where the functor $\underline{\mathrm{Map}}(G, -)$ is induced by Weil-restriction along $\varphi\colon * \to BG$. With $\psi\colon BG \to *$ the canonical map, we then have equivalences

$$\iota \simeq \mathrm{Spec}\,\mathrm{Res}_\psi \qquad\qquad (-)^G \simeq \mathrm{Spec}(-)_G$$
$$U \simeq \mathrm{Spec}\,\mathrm{Res}_\varphi \qquad\qquad (-) \times G \simeq \mathrm{Spec}((-) \otimes \mathcal{O}_G)$$

on affine objects. In general, it also holds that $\iota \simeq \Omega\psi^*$ and $U \simeq \Omega\varphi^*$. From the adjunction between spectra and global sections, together with Proposition 4.10, we moreover have

$$B_G \simeq \mathrm{cInd}_\psi(B) \simeq \Gamma([\mathrm{Spec}\, B/G], \mathcal{O}_{[\mathrm{Spec}\, B/G]})$$

for any $B \in \mathsf{Aff}^G$.

5. Derived loci of maximal stabilizer dimension

Let X be a derived algebraic stack whose classical truncation X_{cl} is Noetherian and admits a good moduli space. In [ER21], it is shown that there is a canonical closed substack $X^{\max}_{\mathrm{cl}}$ of X_{cl} which is the locus of points of X_{cl} with stabilizer group of maximal dimension. Our objective in this section is to establish one of the central results of this paper, the existence of a derived locus of maximal-dimensional stabilizers $X^{\max}$. Our definition of $X^{\max}$ is directly inspired by the definition of $X^{\max}_{\mathrm{cl}}$ in [ER21], and gives a derived enhancement. Along the way we will review derived loop stacks and derived bundles twisted by inner automorphisms, and give a derived Luna étale slice theorem.

5.1. **Derived loop stacks.** Let $X \in \mathsf{Stk}$ be a derived stack. Recall that the *derived loop stack* $\mathscr{L}X$ of X is the self-intersection of the diagonal $\Delta_X \colon X \to X \times X$, which is algebraic if X is. Let $k \colon T \to X$ be a morphism of derived stacks. Then the *stabilizer* of X in T is the pullback

$$G_k \coloneqq G_T \coloneqq T \times_X \mathscr{L}X$$

Since $(\mathscr{L}X)_{\mathrm{cl}}$ is the inertia stack of X_{cl}, it holds that $(G_T)_{\mathrm{cl}}$ is the stabilizer $I_{X_{\mathrm{cl}}}$ of X_{cl} at T_{cl}. We call G_T *connected* if $G_T \to T$ has connected classical fibers. If $\mathcal{O}_T$ is a field, then $k \colon T \to X$ corresponds to a point $x \in X$, and we call $G_x \coloneqq G_k$ the *stabilizer* (at x). Note that $(G_x)_{\mathrm{cl}}$ is a classical affine group scheme.

Example 5.1. The basic example is when X is of the form $[U/G]$, where U is a derived scheme, for which it holds

$$\mathscr{L}X \simeq [S_U/G].$$

Here, S_U is the derived stabilizer group scheme S_U, defined as the fibre product $(G \times U) \times_{U \times U} U$ of the map $(\sigma, \pi_2) \colon G \times U \to U \times U$ (where σ is the G-action) with the diagonal. See [Toë14, §4.4].

Let $f \colon X \to Y$ be a morphism of derived stacks. Then we have a natural morphism $\mathscr{L}f \colon \mathscr{L}X \to \mathscr{L}Y \times_Y X$ of derived stacks over X. We call f *monodromy-free* if $\mathscr{L}f$ is an equivalence. Likewise, we call f *stabilizer-preserving* if $(\mathscr{L}f)_{\mathrm{cl}}$ is an equivalence. If the latter is the case, then for any $k \colon T \to X$ with T classical, the morphism $(\mathscr{L}f)_{\mathrm{cl}}$ induces an equivalence $(G_k)_{\mathrm{cl}} \simeq (G_{fk})_{\mathrm{cl}}$.

We call a morphism of derived algebraic stacks $f \colon X \to Y$ *separated* (*proper*) if f_{cl} is separated (proper), as in [Stcks, Tag 04YW]. If $f \colon X \to Y$ is representable, then f is separated if and only if Δ_f is a closed immersion [Stcks, Tag 04YS]. Hence, since in the general case the diagonal is representable, one can also define f to be separated by asking Δ_f to be of finite type, universally closed, and separated. The basic examples are: affine morphisms are separated [Stcks, Tag 06TZ], and closed immersions are proper [Stcks, Tag 0CL8].

Example 5.2. Let X be a derived algebraic stack. Since we assume that all stacks have affine diagonal, in particular X has separated diagonal. Therefore, for T a separated derived algebraic stack, any morphism $T \to X$ is separated by [Stcks, Tag 050M].

Suppose that we have a Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

of derived algebraic stacks, where the vertical arrows are good moduli spaces. Then if M is separated, so is f.

Remark 5.3. Let $f\colon X \to Y$ be a morphism of derived algebraic stacks. Then we have a Cartesian diagram

$$\begin{tikzcd} \mathscr{L}X \arrow[r] \arrow[d, "\mathscr{L}f"] & X \arrow[d, "\Delta_f"] & \\ f^*\mathscr{L}Y \arrow[r] \arrow[d] & X \times_Y X \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "\Delta_X"] & X \times X \arrow[r] & Y \times Y \end{tikzcd} \tag{5.1}$$

which shows that $\mathscr{L}f$ is a pullback of Δ_f. In particular, if f is representable and separated, then $\mathscr{L}f$ is a closed immersion.

Proposition 5.4. *Let $f\colon X \to Y$ be an étale morphism of derived algebraic stacks such that f_{cl} is stabilizer-preserving. Then f is monodromy-free.*

Proof. Since f is étale, the Cartesian diagram

$$\begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

implies that $X \times_Y X \to X$ is étale. Since the composition $X \xrightarrow{\Delta_f} X \times_Y X \to X$ is the identity, we conclude that Δ_f is étale. By diagram (5.1), since $\mathscr{L}f$ is a pullback of Δ_f, it must also be étale. But $(\mathscr{L}f)_{\mathrm{cl}} = \mathscr{L}(f_{\mathrm{cl}})$ is an isomorphism, since f_{cl} is stabilizer-preserving, so $\mathscr{L}f$ is an equivalence. □

Lemma 5.5. *Suppose that $f\colon X \to Y$ is representable, separated and étale. Then Δ_f, hence $\mathscr{L}f$, is a closed and open immersion.*

Proof. By Remark 5.3, it suffices to show that an étale closed immersion $h\colon S \to T$ is an open immersion. This reduces to the affine case since h is affine, and then to the classical case since h is étale. This last case is known, see [Stcks, Tag 025G]. □

We recall some facts about derived loop stacks from [BZN12]. Let S^1 be the stackification of the presheaf that sends any $T \in \mathsf{Aff}$ to the circle. Then it holds

$$S^1 \simeq B\mathbb{Z} \simeq * \coprod_{* \sqcup *} *$$

where $B\mathbb{Z}$ is the classifying space of the constant group scheme $\mathbb{Z}$. Then for any derived stack X it holds

$$\mathscr{L}X \simeq \underline{\mathrm{Map}}(S^1, X)$$

For any $f\colon X \to Y$ it follows that $f^*\mathscr{L}Y$ is the stack over X which sends $T \to X$ to the space of commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d] \\ S^1 \times T \arrow[r] & Y \end{tikzcd}$$

which we think of as loops on Y based at f. The morphism $\mathscr{L}X \to f^*\mathscr{L}Y$ then sends $T \times S^1 \to X$ to the loop $T \times S^1 \to X \to Y$, which is based at f. In particular, $\mathscr{L}f$ is a morphism of derived group stacks. This also explains the term "monodromy-free".

Remark 5.6. Let T be a derived stack. Throughout, we will write B_T for the functor $B_T\colon \mathrm{Grp}(\mathsf{Stk}_{/T}) \to \mathsf{Stk}_{T/}$ which sends a derived group stack G over T to the colimit of the bar-construction $B(T, G)$ in Stk_T. From Proposition 3.3, it follows that we have an adjunction

$$B_T : \mathrm{Grp}(\mathsf{Stk}_T) \rightleftarrows \mathsf{Stk}_{T/} : \Omega_T$$

where $\Omega_T(T \to X) := \mathscr{L}X \times_X T$, and B_T is fully faithful. This is analogous to the loop space-suspension adjunction from topology.

5.2. **A derived Luna étale slice theorem.** We give a derived version of the étale local structure of algebraic stacks in terms of quotient stacks.

Notation 5.7. For $X \in \mathsf{Stk}$, write $\mathsf{Aff}^{\text{ét}}_X$ for the ∞-category of affine étale morphisms $T \to X$.

Write $|X|$ for the equivalence class of morphisms $x\colon \operatorname{Spec} k \to X$ with k a field, where x is equivalent to $x'\colon \operatorname{Spec} k' \to X$ if there is a common field extension $k, k' \subset K$ such that $x_{|\operatorname{Spec} K} \simeq x'_{|\operatorname{Spec} K}$. Consider the set of points, written $|X|$, as a topological space by declaring $U_0 \subset |X|$ to be open if there is an open immersion $U \to X$ such such that $|U| \to |X|$ has image U_0. Since open immersions are monomorphisms, passing from an open immersion $U \to X$ to the corresponding open subset $|U| \subset |X|$ constitutes an equivalence of partially ordered sets. The induced map $|X_{\mathrm{cl}}| \to |X|$ is a homeomorphism, as the following result shows.

Lemma 5.8 (Topological invariance)**.** *Let X be a quasi-compact derived algebraic stack. Then pulling back along $X_{\mathrm{cl}} \to X$ induces an equivalence $\mathsf{Aff}^{\text{ét}}_X \simeq \mathsf{Aff}^{\text{ét}}_{X_{\mathrm{cl}}}$.*

Proof. The affine case can be found in [GR17, Lem. 2.15].

For the general case, let $U \to X$ be a smooth atlas, where U is a derived scheme. We can assume that U is affine since X is quasi-compact. Then the commutative diagram

$$\begin{tikzcd} U_{\mathrm{cl}} \arrow[r] \arrow[d] & U \arrow[d] \\ X_{\mathrm{cl}} \arrow[r] & X \end{tikzcd}$$

is Cartesian. Now the claim follows from the affine case by descent. □

Suppose that $X \in \mathsf{Stk}$ is such that X_{cl} admits a good moduli space $X_{\mathrm{cl}} \to Y$. We say that a morphism $f\colon [U/G] \to X$ is *strongly étale* if $U \in \mathsf{Aff}^G$, f is étale and f_{cl} fits in a Cartesian diagram

$$\begin{tikzcd} {[U_{\mathrm{cl}}/G]} \arrow[r, "f_{\mathrm{cl}}"] \arrow[d] & X_{\mathrm{cl}} \arrow[d] \\ U_{\mathrm{cl}} /\!\!/ G \arrow[r] & Y. \end{tikzcd}$$

Observe that f_{cl} then has to be stabilizer-preserving and therefore, by Proposition 5.4, also monodromy-free. Moreover, f is representable and separated by Example 5.2, since $U_{\mathrm{cl}}/\!\!/G$ is affine, hence separated. We record these observations in the following proposition.

Proposition 5.9. *Let X be a derived algebraic stack such that X_{cl} admits a good moduli space. Any strongly étale morphism $f\colon [U/G] \to X$ is also representable, separated and monodromy-free.*

We now prove a derived Luna étale slice result.

Proposition 5.10. *Let X be a derived algebraic stack such that X_{cl} is Noetherian and admits a good moduli space. Then X is covered by derived stacks of the form $h\colon [U/G] \to X$, where $U \in \mathsf{Aff}^G$ is such that $U^G \neq \emptyset$, G is reductive, and h is affine and strongly étale (hence monodromy-free by Remark 5.4).*

Proof. Since X_{cl} is Noetherian, the set of closed points is dense in $|X_{\mathrm{cl}}|$. By the Luna étale slice theorem for classical algebraic stacks from [ER21, Thm. A.1], we therefore have a cover of X_{cl} by stacks of the form $h_{\mathrm{cl}}\colon [U_{\mathrm{cl}}/G] \to X_{\mathrm{cl}}$, where U_{cl} is a classical G-scheme such that $U_{\mathrm{cl}}^G \neq \emptyset$, the map h_{cl} is étale, affine and stabilizer-preserving, and G is a stabilizer of some closed point in the image of $U_{\mathrm{cl}} \to X_{\mathrm{cl}}$.

Since X_{cl} is Noetherian, it is quasi-compact. By Lemma 5.8, we thus have a Cartesian diagram

$$\begin{tikzcd} {[U_{\mathrm{cl}}/G]} \arrow[r, "f"] \arrow[d, "h_{\mathrm{cl}}"] & V \arrow[d, "h"] \\ X_{\mathrm{cl}} \arrow[r] & X \end{tikzcd}$$

where the vertical maps are affine and étale, and f_{cl} is an isomorphism. By [Hal20, Lem. 4.2.5], the derived stack V is of the form $[U/G]$, for some derived G-scheme U, and f is induced by a G-equivariant map $f'\colon U_{\mathrm{cl}} \to U$ such that f'_{cl} is an isomorphism.

The collection of morphisms $h\colon [U/G] \to X$, indexed by the elements $h_{\mathrm{cl}}\colon [U_{\mathrm{cl}}/G] \to X_{\mathrm{cl}}$ appearing in the cover coming from the Luna étale slice theorem, is an étale cover, because $|X_{\mathrm{cl}}| \simeq |X|$. □

5.3. **Inner forms.** We introduce ∞-bundles which are twisted versions of a reductive algebraic group G by its adjoint representation G_{ad}, i.e., via inner automorphisms. This will be used to describe $[U/G]^{\max}$ in the cases relevant for the proof of Theorem 5.13.

Definition 5.11. For $f\colon T \to BG$ with corresponding G-bundle E over T, define the *inner form* $\widetilde{G}_f$ of G via the pullback

$$\begin{tikzcd} \widetilde{G}_f \arrow[r] \arrow[d] & \mathscr{L}(BG) \arrow[d] \\ T \arrow[r] & BG. \end{tikzcd} \tag{5.2}$$

Observe that $\widetilde{G}_f$ is a pullback of an ∞-group, hence is an ∞-group itself. In the terminology from [NSS15, Prop. 4.10] it holds that $\widetilde{G}_f$ is the G_{ad}-fiber ∞-bundle $E \times_G G_{\mathrm{ad}}$ associated to E through f, since $\mathscr{L}(BG) = [G_{\mathrm{ad}}/G]$ by definition.

Recall that the identity component G^0 is an open and closed, normal subgroup of G, such that G/G^0 is finite, hence reductive. Let $\pi\colon BG \to B(G/G^0)$. Then we have an induced homomorphism $\varphi\colon \widetilde{G}_f \to \widetilde{G}_{\pi f}$. Since $\widetilde{G}_{\pi f} \to T$ is finite étale, the kernel is an open and closed subgroup that we denote $\widetilde{G}^0_f$. This is a twisted form of G^0 but not necessarily inner.

Lemma 5.12. *Let G be an algebraic group of dimension d, and $\widetilde{G}_f$ an inner form of G classified by $f\colon T \to BG$.*

(i) *$\widetilde{G}_f$ and $\widetilde{G}^0_f$ are derived algebraic group schemes that are smooth of dimension d over T. The group scheme $\widetilde{G}^0_f$ has connected fibers.*

(ii) *$B_T\widetilde{G}_f \simeq T \times BG$ and $B_T(\widetilde{G}^0_f) \simeq T \times_{B(G/G^0)} BG$.*

Proof. (i) Since $\mathscr{L}(BG) \simeq [G_{\mathrm{ad}}/G]$, it holds that $\mathscr{L}(BG)$ is an algebraic group scheme, smooth and fiberwise of dimension d over BG, so by diagram 5.2 the same is true for $\widetilde{G}_f \to T$.

(ii) ∎

5.4. **The derived locus $X^{\max}$ of maximal-dimensional stabilizers.** Throughout this subsection, let X be a derived algebraic stack whose classical truncation X_{cl} admits a good moduli space Y. We moreover assume that X_{cl} is Noetherian. Recall that all stacks have affine diagonal by assumption. In what follows, $G^0 \subseteq G$ denotes the identity component of any (possibly disconnected) algebraic group G.

The goal is to show the following.

Theorem 5.13. *Let d be the maximal dimension of stabilizers of points of X. Then there exists a canonical closed immersion $X^{\max} \to X$ with the following property: For any affine, strongly étale morphism $[U/G] \to X$, where $U \in \mathsf{Aff}^G$ and G is reductive with $\dim G = d$, there exists a Cartesian square*

$$\begin{tikzcd} {[U^{G^0}/G]} \arrow[r] \arrow[d] & {[U/G]} \arrow[d] \\ X^{\max} \arrow[r] & X. \end{tikzcd}$$

Then $(-)_{\mathrm{cl}}$ commutes with $(-)^{\max}$, and $X^{\max}_{\mathrm{cl}}$ is equivalent to the classical locus of point of X_{cl} with maximal stabilizer dimension defined in [ER21, Prop. C.5].

Since $|X_{\mathrm{cl}}| \simeq |X|$, the map $x \mapsto \dim G_x$ is upper semi-continuous on $|X|$ by the same argument as for the classical case, see [ER21]. We let $X^{\leq d} \to X$ be the open immersion corresponding to the open subset of $|X|$ consisting of points x with $\dim G_x \leq d$. Observe, if X is classical then so is $X^{\leq d}$, and in general it holds $(X^{\leq d})_{\mathrm{cl}} \simeq (X_{\mathrm{cl}})^{\leq d}$.

Definition 5.14. For $d \in \mathbb{N}$, let X^d be the stack over X such that $X^d(T \to X)$ is the space of closed derived subgroup schemes $H^0 \to T \times_X \mathscr{L}X$ which are smooth over T with connected fibers of dimension d. Naturality in T is defined by pullback. If d is the maximal dimension of stabilizers of X, we define $X^{\max} := X^d$.

Remark 5.15. Let $T \in \mathsf{Stk}_X$ be of the form $[V/G]$ and X of the form $[U/G]$, where U, V are affine derived G-schemes, and G is reductive. Then it holds that $X^d(T \to [U/G])$ is the space of derived closed subgroup stacks $H^0 \to T \times_X \mathscr{L}X$, such that after pulling back along $q\colon V \to [V/G]$ it holds that q^*H^0 is a derived closed subgroup scheme smooth over V with connected fibers of dimension d.

In order to prove several properties satisfied by $X^{\max}$, we will need a few auxiliary results.

Proposition 5.16. *Suppose that X is of the form $[U/G]$ for an affine derived scheme U and a reductive group G. Then the derived stack $[U^{G^0}/G]$ has functor of points*

$$[U^{G^0}/G](f\colon T \to X) \simeq \mathsf{Stk}_{T/\cdot/BG}(T \times_{B(G/G^0)} BG, X),$$

where $\mathsf{Stk}_{T/\cdot/BG}$ is the double slice category $(\mathsf{Stk}_{T/})_{T \to BG}$.

Proof. Write $\pi\colon BG \to B(G/G^0)$ and $p\colon X = [U/G] \to BG$ for the projections. Let $f\colon T \to X$ be given, and let $\pi_!$ be the left adjoint of π^*, the functor which sends $S \to BG$ to $S \to B(G/G^0)$. Then it holds, using the universal property of the Weil restriction,

$$\begin{aligned}\mathsf{Stk}_X(T, \pi^*\pi_*X) &\simeq \mathsf{Stk}_{BG}(T, \pi^*\pi_*X) \times_{\mathsf{Stk}_{BG}(T,X)} \{f\}\\ &\simeq \mathsf{Stk}_{BG}(\pi^*\pi_!T, X) \times_{\mathsf{Stk}_{BG}(T,X)} \{f\}\\ &\simeq \mathsf{Stk}_{T/\cdot/BG}(T \times_{B(G/G^0)} BG, X). \qquad \square\end{aligned}$$

Let $p\colon X \to BG$ and $f\colon T \to X$ be given. Then we have a commutative diagram

$$\begin{array}{ccc} f^*\mathscr{L}(X) & \longrightarrow & \mathscr{L}(BG) \\ \downarrow & & \downarrow \\ T & \xrightarrow{\;\;pf\;\;} & BG \end{array}$$

and thus a morphism $h\colon f^*\mathscr{L}(X) \to \widetilde{G}_{pf}$ over T by definition of inner forms. In what follows, we will consider $f^*\mathscr{L}(X)$ as derived stack over $\widetilde{G}_{pf}$ via h.

Proposition 5.17. *Suppose that X is of the form $[U/G]$ for an affine derived scheme U and a reductive group G of dimension d, write $p\colon [U/G] \to BG$ for the projection. Then the derived stack X^d has functor of points*

$$X^d(f\colon T \to X) \simeq \mathrm{Grp}(\mathsf{Stk}_{\widetilde{G}_{pf}})(\widetilde{G}^0_{pf}, f^*\mathscr{L}X).$$

Proof. Recall that $\widetilde{G}^0_{pf} \to \widetilde{G}_{pf}$ is a closed and open subgroup which is smooth with connected fibers over T. Since $f^*\mathscr{L}X \to \widetilde{G}_{pf}$ is a closed subgroup, it follows that any homomorphism $\widetilde{G}^0_{pf} \to f^*\mathscr{L}X$ is a closed immersion. We thus obtain a natural map

$$\varphi\colon \operatorname{Grp}(\mathsf{Stk}_{\widetilde{G}_{pf}})(\widetilde{G}^0_{pf}, f^*\mathscr{L}X) \to X^d(f\colon T \to X) \tag{5.3}$$

which we claim is an equivalence.

Let $H^0 \to f^*\mathscr{L}X$ be any closed derived subgroup scheme, smooth over T and with connected fibers of dimension d. Consider the diagram

$$\begin{tikzcd} & & \widetilde{G}^0_{pf} \arrow[d, "\beta"] \\ H^0 \arrow[r] \arrow[urr, dotted, "\alpha"] & f^*\mathscr{L}X \arrow[r] & f^*p^*\mathscr{L}(BG) = \widetilde{G}_{pf} \end{tikzcd}$$

Since β is a closed and open immersion, the existence of the dotted arrow can be checked on classical truncations, on which this holds by [ER21, Appendix C], and on which moreover it is an isomorphism [ibid.]. Since β is a monomorphism, the arrow α is canonically unique and satisfies that α_{cl} is an isomorphism. Since $H^0, \widetilde{G}^0_{pf}$ are flat over T, also α is invertible. Clearly, the inverse of α induces a morphism of derived group stacks $\widetilde{G}^0_{pf} \to f^*\mathscr{L}X$ over $\widetilde{G}_{pf}$, which shows that φ is surjective on π_0.

By our argument above, the fiber of φ over any point is the space of automorphisms of the derived group stack $\widetilde{G}^0_{pf}$ over $\widetilde{G}_{pf}$, which is contractible, since the morphism $\widetilde{G}^0_{pf} \to \widetilde{G}_{pf}$ is a monomorphism. This concludes the proof. □

Theorem 5.13 will now be an immediate consequence of the following.

Proposition 5.18. *Let also Y be a derived algebraic stack such that Y_{cl} is Noetherian and admits a good moduli space.*

(i) *$|X^d| \to |X|$ is injective, and identifies $|X^d|$ with the subset $|X|^d \subset |X|$ of points with stabilizer dimension d.*
(ii) *The construction $(-)^d$ is functorial in representable and separated morphisms.*
(iii) *If $g\colon X \to Y$ is representable, separated and étale, then the natural map $X^d \to X \times_Y Y^d$ is an equivalence.*
(iv) *Suppose that X is of the form $[U/G]$ for an affine derived scheme U and a d-dimensional reductive group G. If G^0 acts trivially on U, then $[U/G]^d \to [U/G]$ is an equivalence.*
(v) *In general, for $X = [U/G]$ as in the previous point, the morphism $[U^{G^0}/G] \to [U/G]$ induces an equivalence $[U^{G^0}/G] \simeq [U/G]^d$.*
(vi) *The derived stack X^d is algebraic.*
(vii) *The canonical map $(X^d)_{\mathrm{cl}} \to X_{\mathrm{cl}}$ is the closed substack described in* [ER21, Appendix C].
(viii) *$X^d \to X$ factors through $X^{\le d}$ via a (necessarily unique) closed immersion $X^d \to X^{\le d}$.*

Proof. (i) Let $x\colon \operatorname{Spec} k \to X$ be given. Then

$$(G_x)^0_{\mathrm{cl}} \subset G_x = \mathscr{L}(X) \times_X \operatorname{Spec} k$$

is the unique closed subgroup of G_x of dimension $d' := \dim G_x$ which is smooth and connected. From this, (i) follows.

(ii) Let $g\colon X \to Y$ be representable and separated. For any $T \to X$, we have a Cartesian square

$$\begin{tikzcd} T \times_X \mathscr{L}X \arrow[r] \arrow[d] & \mathscr{L}X \arrow[d, "\mathscr{L}g"] \\ T \times_Y \mathscr{L}Y \arrow[r] & X \times_Y \mathscr{L}Y, \end{tikzcd} \tag{5.4}$$

hence $T \times_X \mathscr{L}X \to T \times_Y \mathscr{L}Y$ is a closed immersion by Remark 5.3. The claim follows.

(iii) Assume moreover that $g\colon X \to Y$ is étale. Then $\mathscr{L}g\colon \mathscr{L}X \to X \times_Y \mathscr{L}Y$ is a closed and open immersion by Lemma 5.5. From the diagram (5.4), it follows that $\varphi\colon T \times_X \mathscr{L}X \to T \times_Y \mathscr{L}Y$ is an open and closed immersion, hence a monomorphism, for any $f\colon T \to X$. Let a T-point of Y^d be classified by $\alpha\colon H^0_Y \to T \times_Y \mathscr{L}Y$, and let $H^0_X \to T \times_X \mathscr{L}X$ be the pullback of α along φ. Then the projection $\gamma\colon H^0_X \to H^0_Y$ is an open and closed immersion over T, which is an equivalence since $H^0_Y \to T$ has connected fibers. We have thus described a T-point of X^d such that composing with φ recovers α.

(iv) Let $X = [U/G]$ with U affine and $\dim G = d$, and suppose that G^0 acts trivially on U. Let S_U be the derived stabilizer group scheme $(G\times U)\times_{U\times U} U$ as in Example 5.1. Since G^0 acts trivially on U, we have a natural map $G^0 \times U \to S_U$ over $G \times U$, and thus morphisms

$$[G^0 \times U/G] \xrightarrow{\beta} \mathscr{L}X \xrightarrow{\alpha} [U/G]$$

over $[U/G]$. Since $* \to BG$ is smooth, so is $U \to [U/G]$. Likewise, $G^0 \times U \to U$ is smooth, as it is the pullback of the smooth map $G^0 \to *$ along $U \to *$. It follows that $\alpha\beta$ is smooth. Clearly, the fibers of $\alpha\beta$ are connected and of dimension d. Moreover, since U is affine, the morphism $S_U \to G \times U$ is separated, hence $G^0 \times U \to S_U$ is a closed immersion, since $G^0 \times U \to G \times U$ is. By Remark 5.15, we therefore have a morphism $\varphi\colon [U/G] \to [U/G]^d$ over $[U/G]$. To see that it is an equivalence, write $\chi\colon [U/G]^d \to [U/G]$ for the natural map. By construction, it holds $\chi\varphi \simeq \mathrm{id}_{[U/G]}$.

To show that also $\varphi\chi \simeq \mathrm{id}_{[U/G]^d}$, let $F^0 \to \chi^*\mathscr{L}X$ be the derived closed subgroup stack classifying the identity on $[U/G]^d$ as stacks over $[U/G]$. Then F^0 is universal in the following sense: for a T-point $f \in X(T)$, we have that any $f' \in X^d(T)$, say classified by $H^0 \to f^*\mathscr{L}X$, trivially satisfies $\chi f' \simeq f$, and hence that $H^0 \simeq f'^*F^0$ by definition of $(-)^d$. In particular, for $\psi\colon [U/G]^d \to [U/G]^d$ any map over $[U/G]$, since $\chi\psi \simeq \chi$, it follows that ψ is classified by F^0 as well, hence $\psi \simeq \mathrm{id}_{[U/G]^d}$. Since $\chi\varphi\chi \simeq \chi$, the claim follows by applying this observation to $\psi := \varphi\chi$.

(v) Let still $X = [U/G]$, put $Y := [U^{G^0}/G]$, and write $p\colon X \to BG$ for the projection. Since $Y \to X$ is representable and separated, we have a natural map $Y^d \to X^d$ by (ii). Also, since G^0 acts trivially on U^{G^0}, by (iv) the structure map $Y^d \to Y$ is an equivalence. We therefore get a natural morphism $h\colon Y \to X^d$ over X, which we claim to be an equivalence.

For any map $f\colon T \to X$, using Propositions 5.16 and 5.17, the morphism $h(T) : Y(T) \to X^d(T)$ is equivalent to

$$\mathsf{Stk}_{T/\cdot/BG}(T \times_{B(G/G^0)} BG, X) \xrightarrow{h(T)} \mathrm{Grp}(\mathsf{Stk}_{\widetilde{G}_{pf}})(\widetilde{G}^0_{pf}, f^*\mathscr{L}X). \tag{5.5}$$

We need to check that this morphism is an equivalence. Recall, in the adjunction $B_T \dashv \Omega_T$ from Remark 5.6, it holds that B_T is fully faithful. We therefore have

$$\widetilde{G}^0_{pf} \simeq \Omega_T B_T \widetilde{G}^0_{pf} \simeq \Omega_T(T \times BG^0)$$

by Lemma 5.12. By the same lemma and by adjunction, it follows that

$$Y(T) \simeq \mathsf{Stk}_{T/\cdot/BG}(B_T\widetilde{G}^0_{pf}, X) \simeq \mathrm{Grp}(\mathsf{Stk}_T)_{/\widetilde{G}_{pf}}(\widetilde{G}^0_{pf}, \Omega_T X) \tag{5.6}$$

Again by Remark 5.6, we have

$$\mathrm{Grp}(\mathsf{Stk}_T)_{/\widetilde{G}_{pf}}(\widetilde{G}^0_{pf}, \Omega_T X) \simeq \mathrm{Grp}(\mathsf{Stk}_{\widetilde{G}_{pf}})(\widetilde{G}^0_{pf}, f^*\mathscr{L}X) \simeq X^d(T)$$

Combining with the equivalence in (5.6) shows that (5.5) is an equivalence, which finishes the argument.

(vi) By the étale local structure of X (Propositions 5.10 and 5.9), we can reduce to the case where $X = [U/G]$ by (iii). The statement then follows from (v).

(vii) Let $f\colon T \to X$ be given with T a classical scheme, so that we have a unique factorization $f\colon T \to X_{\mathrm{cl}} \to X$. The space $(X^d)_{\mathrm{cl}}(T)$ thus classifies derived closed subgroup schemes $H^0 \to T \times_X \mathscr{L}X$ over T. Since T is classical and $H^0 \to T$ is smooth, we must have $H^0 = (H^0)_{\mathrm{cl}}$, and $H^0 \to T \times_X \mathscr{L}X$ is naturally equivalent to the closed embedding of group schemes $H^0 \to (T \times_X \mathscr{L}X)_{\mathrm{cl}} = T \times_{X_{\mathrm{cl}}} (\mathscr{L}X)_{\mathrm{cl}} = T \times_{X_{\mathrm{cl}}} \mathscr{L}X_{\mathrm{cl}}$. These are the T-points of $(X_{\mathrm{cl}})^d$ by [ER21, Prop. C.5].

(viii) Since $X^{\leq d} \to X$ is an open immersion, the factorization exists uniquely by (i). To check that the induced map $X^d \to X^{\leq d}$ is a closed immersion, we reduce to the classical case by (vii). By the argument for (vii), the statement then follows from [ER21, Prop. C.5]. □

Proof of Theorem 5.13. Everything follows from Proposition 5.10 and Proposition 5.18. □

6. Intrinsic blow-ups as equivariant derived blow-ups

Let U be a classical scheme with an action by a reductive group G. The main aim of this section is to prove that the G-intrinsic blow-up of U, introduced in [KLS17], has a natural derived enhancement which is obtained by a corresponding derived blow-up construction, namely blowing up U along its derived fixed locus U^G.

To this end, we begin with a review of intrinsic blow-ups and then proceed to establish the main result comparing intrinsic and derived blow-ups by using standard equivariant local models for U. For simplicity, throughout, we initially assume that G is connected and then explain how to adapt the constructions in the non-connected case. We conclude with some remarks and a few instructive examples.

The operations of intrinsic blow-up and its derived enhancement will be fundamental building blocks in our derived stabilizer reduction procedure in Section 7.

6.1. **Background on intrinsic blowups.** Let U be a classical affine scheme with an action of a reductive group G. Assume for simplicity that G is connected.

Take an equivariant closed embedding $U \to V$ into a smooth G-scheme V and let I be the ideal defining U. Since $U \subseteq V$ is G-equivariant, G acts on I and we have a decomposition $I = I^{\mathrm{fix}} \oplus I^{\mathrm{mv}}$ into the fixed part of I and its complement as G-representations.

Let V^G be the fixed point locus of G inside V, defined by the ideal generated by $\mathcal{O}_V^{\mathrm{mv}}$ and $\pi\colon \mathrm{bl}_G(V) \to V$ the blowup of V along V^G. Let $E \subseteq \mathrm{bl}_G(V)$ be its exceptional divisor and $\xi \in \mathcal{O}_{\mathrm{bl}_G(V)}(E)$ the tautological defining equation of E.[1]

G-equivariance implies that (cf. [KLS17, Section 2.2])

$$\pi^{-1}(I^{\mathrm{mv}}) \subseteq \xi \cdot \mathcal{O}_{\mathrm{bl}_G(V)}(-E) \subseteq \mathcal{O}_{\mathrm{bl}_G(V)},$$

and consequently, $\xi^{-1}\pi^{-1}(I^{\mathrm{mv}}) \subseteq \mathcal{O}_{\mathrm{bl}_G(V)}$, meaning that $\pi^{-1}(I^{\mathrm{mv}})$ lies in the image of the inclusion $\mathcal{O}_{\mathrm{bl}_G(V)}(-E) \subseteq \mathcal{O}_{\mathrm{bl}_G(V)}$, given by multiplication by ξ.

We define $I^{\mathrm{intr}} \subseteq \mathcal{O}_{\mathrm{bl}_G(V)}$ to be the ideal sheaf

$$I^{\mathrm{intr}} = \text{ideal sheaf generated by } \pi^{-1}(I^{\mathrm{fix}}) \text{ and } \xi^{-1}\pi^{-1}(I^{\mathrm{mv}}). \tag{6.1}$$

Definition 6.1 ([KLS17])**.** The G-intrinsic blowup of U (respectively $[U/G]$) is the G-invariant subscheme $U^{\mathrm{intr}} \subseteq \mathrm{bl}_G(V)$ (respectively substack $[\widehat{U}/G]$) defined by the ideal I^{intr}.

In [KLS17], it is proved by hands-on computation that U^{intr} is intrinsic to the G-scheme U. Moreover, by [KLS17, Sav20], the construction globalizes to any scheme U with a G-action and a Zariski open cover by G-invariant subschemes.

Example 6.2. Suppose that $G = \mathbb{C}^\times$ is the one-dimensional torus acting on the affine plane $V = \mathbb{C}^2_{x,y}$ with weights 1 and -1 on the coordinates x and y respectively and $U \subseteq V$ is the closed G-invariant subscheme cut out by the ideal $I = (x^2y, xy^2)$.

V^{intr} is the blowup of V along the fixed locus $V^G = \{0\}$ cut out by the ideal $\mathcal{O}_V^{\mathrm{mv}} = (x, y)$. Writing u, v for the homogeneous coordinates on the exceptional divisor $\mathbb{P}^1$, we have that G acts linearly on u, v with weights $1, -1$ respectively and the blowdown map $V^{\mathrm{intr}} \to V$ is locally given on coordinates by $x \mapsto \xi u, y \mapsto \xi v$.

Since $I^{\mathrm{mv}} = (x^2y, xy^2)$, $U^{\mathrm{intr}} \subseteq V^{\mathrm{intr}}$ is the closed subscheme locally cut out by the ideal $I^{\mathrm{intr}} = (\xi^2u^2v, \xi^2uv^2)$.

[1] Recall, for a divisor $D \subseteq X$, one writes $\mathcal{O}_X(D)$ for the fractional ideal sheaf locally generated by f^{-1}, where f is such that locally $D = V(f)$. For a closed immersion $Z \to X$ with ideal I, the blow-up of X in Z is locally of the form $\mathrm{Spec}(\mathcal{O}_X[It, t^{-1}]_{ft})$, and so the tautological defining equation is $t^{-1} = f/(ft)$.

We now briefly explain how one can proceed if G is not connected.

Suppose that $U \to V$ is a G-equivariant embedding into a smooth G-scheme V. Let, as before, I be the ideal of U in V. Let G^0 be the connected component of the identity. This is a normal, connected subgroup of G of finite index. Let $I = I^{\text{fix}} \oplus I^{\text{mv}}$ be the decomposition of I into fixed and moving parts with respect to the action of G^0. Using the normality of G^0, we see that the fixed locus V^{G^0} is a closed, smooth G-invariant subscheme of V and also I^{fix}, I^{mv} are G-invariant.

Let $\pi\colon \mathrm{bl}_{V^{G^0}} V \to V$ be the blow-up of V along V^{G^0} with exceptional divisor E and local defining equation ξ. Then, as before, take I^{intr} to be the ideal generated by $\pi^{-1}(I^{\text{fix}})$ and $\xi^{-1}\pi^{-1}(I^{\text{mv}})$. Everything is G-equivariant and we define U^{intr} as the subscheme of $\mathrm{bl}_{V^{G^0}} V$ defined by the ideal I^{intr}. At the level of quotient stacks, $[U^{\text{intr}}/G]$ is the intrinsic blow-up of $[U/G]$.

6.2. **Intrinsic blow-ups are classical truncations of derived blow-ups.** Consider a derived affine scheme U, quasi-compact and locally of finite presentation, with an action by a connected reductive group G such that the (derived) fixed locus $(U_{\text{cl}})^G$ is non-empty. The following is the main result of this section.

Theorem 6.3. *The G-intrinsic blowup of the classical truncation U_{cl} is naturally isomorphic, as a derived scheme with G-action, to the classical truncation $(\mathrm{Bl}_{U^G} U)_{\text{cl}}$ of the derived blow-up of U along the fixed locus U^G.*

Proof. The statement is local. Let $x \in (U_{\text{cl}})^G$.

By Lemma 4.25, after possibly (Zariski) equivariantly shrinking U around x, we may reduce to the case where $U = \operatorname{Spec} R'$ and R' corresponds to a cdga in standard form (with homological indexing)

$$R = [\ldots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow R_0], \tag{6.2}$$

where $R_0 = \mathbb{C}[x_1, ..., x_N]$, G acts linearly on the variables x_i, and R' is generated in each positive degree by a G-representation V^i with variables y^i_j corresponding to a basis for each V^i.

Let $V = \operatorname{Spec} R_0$. We then know that $U^G = \operatorname{Spec} S$, with $S^0 = \mathcal{O}_{V^G}$, and generated in each positive degree i by the G-fixed variables of V^i.

Consider now the auxiliary scheme $F_1 = \operatorname{Spec} S_1$, where $S_1^0 = \mathcal{O}_{V^G}$ and S_1 is generated in degree 1 by the G-fixed variables of V^1 and by all of the variables of V^i for $i \geq 2$.

There is a composition $U^G \to F_1 \to U$, where the morphism $U^G \to F_1$ is easily seen to be a sequence of cell attachments in degrees bigger than 1, using the fact that $(V^2)^{\text{mv}}$ maps to 0 in S_{11} by G-equivariance.

In particular, Proposition 3.10 gives an isomorphism

$$(\mathrm{Bl}_{U^G} U)_{\text{cl}} \simeq (\mathrm{Bl}_{F_1} U)_{\text{cl}}, \tag{6.3}$$

and we are thus reduced to calculating $(\mathrm{Bl}_{F_1} U)_{\text{cl}}$.

Let $A \subseteq R$ be the subalgebra of R generated by R^0 and the G-fixed variables in degree 1.

Without loss of generality, let $x_1, ..., x_l$ be the moving variables in R_0 and $f_1, ..., f_n \in (x_1, ..., x_l) = (R_0^{\text{mv}})$ the images of the moving variables in V^{-1}.

Set
$$B = A/(f_1, ..., f_n),\ D = A/(x_1, ..., x_l)$$
with the obvious cell attachment maps.

Let $Z = \operatorname{Spec} D,\ X = \operatorname{Spec} B,\ Y = \operatorname{Spec} A$. It is clear that B is the subalgebra of R generated by R_0 and R_1 and we have a morphism $U \to X$.

We then have $Z_{\mathrm{cl}} = (U_{\mathrm{cl}})^G$, $X_{\mathrm{cl}} = U_{\mathrm{cl}}$ and a Cartesian square

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r] & X. \end{tikzcd} \tag{6.4}$$

Proposition 3.11 then implies that

$$(\mathrm{Bl}_{F_1} U)_{\mathrm{cl}} \simeq (\mathrm{Bl}_Z X)_{\mathrm{cl}}. \tag{6.5}$$

But, by (3.3), it immediately follows that $(\mathrm{Bl}_Z X)_{\mathrm{cl}}$ is the G-intrinsic blowup of $X_{\mathrm{cl}} = U_{\mathrm{cl}}$, as we want, obtained using the equivariant closed embedding $U_{\mathrm{cl}} \to V$.

Indeed, $V^G = \operatorname{Spec} R_0/(x_1, \ldots, x_l)$, since $x_1, \ldots, x_l$ are the moving variables among the x_i.

Letting I be the ideal of the canonical inclusion $U_{\mathrm{cl}} \to V$, I is then the image of R_1, hence $I^{\mathrm{mv}} = (f_1, \ldots, f_n)$. Write $f_i = \sum_{ij} \lambda_{ij} x_j$. Thus in the Rees algebra
$$R^{G,\mathrm{ext}}_{V^G/V} = \frac{R_0[t^{-1}, w_1, \ldots, w_l]}{(t^{-1}w_1 - x_1, \ldots, t^{-1}w_l - x_l)}$$
it holds that $\xi^{-1}\pi^{-1} f_j = \sum_{ij} \lambda_{ij} w_j$, where $\pi\colon \mathrm{bl}_{V^G} V \to V$ is the projection. Likewise, $\pi^{-1}(I^{\mathrm{fix}})$ is generated by $\partial(V^1)^{\mathrm{fix}}$ and hence equals the image ∂A_1, where ∂ is the boundary map.

We have that the intrinsic blow-up of U_{cl} is the projective spectrum of the discrete quotient
$$\frac{R_0[t^{-1}, w_1, \ldots, w_l]}{(t^{-1}\underline{w} - \underline{x}, \partial A_1, \sum_{1j} \lambda_{ij} w_j, \ldots, \sum_{nj} \lambda_{nj} w_j)} \simeq \frac{\pi_0(A)[t^{-1}, \underline{w}]}{(t^{-1}\underline{w} - \underline{x}, \sum_{ij} \lambda_{ij} w_j)}$$
which we recognize as $\pi_0 R^{G,\mathrm{ext}}_{Z/X}$, using (3.3). □

A special case of the theorem which is worth noting is when U is classical to begin with.

Corollary 6.4. *Let U be a classical affine scheme, of finite type and locally of finite presentation as a derived scheme, with a G-action. Then its G-intrinsic blow-up U^{intr} is the classical truncation of the derived blow-up $\mathrm{Bl}_{U^G} U$ of U with center the **derived** fixed locus U^G.*

Remark 6.5. Even though the above corollary is stated for a classical scheme of finite type and of local finite presentation as a derived scheme, this is not a restriction, and it still holds in the more general case of a scheme of finite type. This is because the proof of Lemma 4.25 for a classical G-scheme of finite type shows that one can obtain a local model $U = \operatorname{Spec} R$ as a possibly infinite sequence of equivariant cell attachments, which are finitely many in each homological degree, i.e., a possibly infinite equivariant standard form. This, however, does not affect the computations, as is clear from our argument above. We elect to assume local finite presentation for simplicity. The same applies to Section 7.

Remark 6.6. Since U^G is well-defined, it is now conceptually clear why the intrinsic blow-up only depends on the G-scheme U and not the auxiliary data used to define it, which is a priori not obvious and had to be checked by hand in [KLS17].

Another non-obvious consequence of the above results is the fact that the classical truncation of the blow-up $\mathrm{Bl}_{U^G} U$ for a derived G-scheme U only depends on its classical underlying G-scheme U_{cl}.

6.3. **A couple of examples.** We give two examples which hopefully clarify the arguments in this section and will also be useful guides for intuition. Both are derived enhancements of Example 6.2 and we can directly verify that the classical truncation of the derived blow-up in each of them coincides with the intrinsic blow-up described in Example 6.2.

Example 6.7. Put $A := \mathbb{C}[x, y]$, and let V be $\operatorname{Spec} A$. Consider $f = x^2y^2$ as a map $V \to \mathbb{A}^1_{\mathbb{C}}$. Then $df\colon \mathcal{O}_V \to \Omega_V$ is the map that sends x to $2xy^2$ and y to $2yx^2$.

The derived critical locus of f is then defined as the intersection of $df\colon V \to \underline{\Omega_V}$ with the zero-section. This is equal to the intersection of the map $V \to V$ that sends (x, y) to (xy^2, yx^2) with the map $\{0\} \to V$. Write U for this intersection, so that $U = \operatorname{Spec} B$, where $B := \mathbb{C}[x, y]/(xy^2, yx^2)$.

Write $Z = \operatorname{Spec} \mathbb{C}$, and consider the closed embedding $Z \to X$ given by the quotient map $B \to A/(x, y) \simeq \mathbb{C}$. Then Lemma 3.12 gives us that $D_{Z/U} \simeq D_{Z/V} \times_{D_{U/V}} D_{U/U}$. Hence, in terms of Rees algebras, we have

$$R^{\mathrm{ext}}_{Z/U} \simeq B[t^{-1}] \otimes_{A[t^{-1},u,v]/(t^{-1}u - x^2y,\, t^{-1}v - y^2x)} A[It, t^{-1}] \tag{6.6}$$

where I is the ideal (x, y), and u, v are homogeneous in degree 1.

Now consider the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z}[p,q] \arrow[r, "{p,q \mapsto 0}"] \arrow[d, "f"] & \mathbb{Z} \arrow[d] & \\
A[t^{-1}, u, v] \arrow[r] \arrow[d, "{u,v \mapsto 0}"] & \frac{A[t^{-1},u,v]}{(t^{-1}u - x^2y,\, t^{-1}v - y^2x)} \arrow[r, "g"] \arrow[d, "{u,v \mapsto 0}"] & A[It, t^{-1}] \arrow[d] \\
A[t^{-1}] \arrow[r] & \frac{A[t^{-1}]}{(xy^2, yx^2)} \arrow[r] & \mathcal{R}^{\mathrm{ext}}_{Z/U}
\end{tikzcd}$$

where $f(p) = t^{-1}u - x^2y, f(q) = t^{-1}v - y^2x$ and $g(u) = xy^2t, g(v) = yx^2t$, and the bottom right square is the coCartesian diagram exhibiting (6.6). Observe that the two squares on the left together form a pushout diagram, hence the square on the bottom left is a pushout. Hence the bottom two squares together are a pushout, and thus

$$R^{\mathrm{ext}}_{Z/U} \simeq A[It, t^{-1}]/(xy^2t, yx^2t).$$

Example 6.8. As before, let $A = \mathbb{C}[x,y]$, $B = A/(xy^2, yx^2)$ and $Z = \operatorname{Spec}\mathbb{C}$, $U = \operatorname{Spec} B$.

The Koszul complex defining the algebra B looks like $A \to A \oplus A \to A$, which gives us an element $(x, -y) \in A \oplus A$ that gives rise to a nonzero element

$$\sigma \in \pi_1 B \simeq \pi_0 \operatorname{Map}(\operatorname{Sym}(B[1]), B)$$

and thus a map $f_\sigma\colon \operatorname{Sym}(B[1]) \to B$. Let C be the pushout of f_σ along the map $\operatorname{Sym}(B[1]) \to B$ induced by the zero-map $B[1] \to B$, so that $C = B/(\sigma)$.

Put $W = \operatorname{Spec} C$. Using the previous example, we will compute $R^{\mathrm{ext}}_{Z/W}$.

As before, we have

$$D_{Z/W} \simeq D_{Z/U} \times_{D_{W/X}} D_{W/W}$$

Now consider the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z}[w] \arrow[r, "w\mapsto vt^{-1}-\sigma"] \arrow[d, "w\mapsto 0"] & B[t^{-1}, v] \arrow[r, "v\mapsto 0"] \arrow[d] & B[t^{-1}] \arrow[d] \\
\mathbb{Z} \arrow[r] & \frac{B[t^{-1},v]}{(vt^{-1}-\sigma)} \arrow[r] \arrow[d, "v\mapsto\tau"] & C[t^{-1}] \arrow[d] \\
 & \mathcal{R}^{\mathrm{ext}}_{Z/U} \arrow[r] & \mathcal{R}^{\mathrm{ext}}_{Z/W}
\end{tikzcd}$$

where v, w are free in simplicial level 1 and homogeneous degree 1, and τ is the element corresponding to $(xt, -yt)$ in degree one of the Koszul complex $R \to R \oplus R \to R$ of $\mathcal{R}^{\mathrm{ext}}_{Z/U}$, where $R = A[It, t^{-1}]$ for $I = (x, y)$, and all squares are coCartesian. This shows that

$$R^{\mathrm{ext}}_{Z/W} \simeq R^{\mathrm{ext}}_{Z/U}/(\tau).$$

7. Derived stabilizer reduction of Artin stacks

Let X be a derived Artin stack, of finite type and locally of finite presentation, with classical truncation X_{cl}. Throughout this section we assume that X_{cl} has affine diagonal and admits a good moduli space $X_{\mathrm{cl}} \to M$ with affine diagonal.

Under these assumptions, an intrinsic stabilizer reduction procedure was developed in [KLS17, Sav20], producing a canonical sequence of classical Artin stacks

$$X^0_{\mathrm{cl}} = X_{\mathrm{cl}},\ X^1_{\mathrm{cl}} := \widehat{X}^0_{\mathrm{cl}},\ \ldots,\ X^m_{\mathrm{cl}} := \widehat{X}^{m-1}_{\mathrm{cl}} \tag{7.1}$$

where each X^i_{cl} is obtained by applying an operation called Kirwan blow-up on the preceding stack X^{i-1}_{cl} (denoted by the top hat) and the maximal stabilizer dimension of the stacks in the sequence is strictly decreasing, so that X^m_{cl} has only finite stabilizers. The Deligne–Mumford stack $\widetilde{X}_{\mathrm{cl}} := X^m_{\mathrm{cl}}$ is called the intrinsic stabilizer reduction of X_{cl}.

In this section, we use the results obtained so far to upgrade this construction to the derived context. More precisely, we define a derived Kirwan blow-up operation whose classical truncation is the classical Kirwan blow-up. This allows us to construct a derived stabilizer reduction procedure by a canonical sequence of derived Artin stacks

$$X^0 = X,\ X^1 := \widehat{X}^0,\ \ldots,\ X^m := \widehat{X}^{m-1} \tag{7.2}$$

whose classical truncation is (7.1). We will define the derived Deligne–Mumford stack $\widetilde{X} := X^m$ to be the derived stabilizer reduction of X.

7.1. **Review of the classical case.** We briefly recall the notions of intrinsic and Kirwan blow-up for a classical Artin stack X_{cl} with a good moduli space $q\colon X_{\mathrm{cl}} \to M$, as considered in [Sav20]. Let $M^{\max}$ be the good moduli space of $X_{\mathrm{cl}}^{\max}$.

The classical intrinsic blow-up

$$\pi^{\mathrm{intr}}\colon X_{\mathrm{cl}}^{\mathrm{intr}} \to X_{\mathrm{cl}}$$

of X_{cl} is defined by étale descent using affine, strongly étale morphisms $[U_{\mathrm{cl}}/G] \to X_{\mathrm{cl}}$, where G runs along all reductive stabilizer groups of X_{cl} of maximum dimension. Given such, $X_{\mathrm{cl}}^{\mathrm{intr}}$ is obtained through a corresponding affine, étale, stabilizer-preserving cover by morphisms

$$[U_{\mathrm{cl}}^{\mathrm{intr}}/G] \to X_{\mathrm{cl}}^{\mathrm{intr}} \tag{7.3}$$

glued together with the complement $X_{\mathrm{cl}} \setminus X_{\mathrm{cl}}^{\max} \subseteq X_{\mathrm{cl}}$ of $X^{\max}$, which is unaffected by the intrinsic blow-ups of the cover.

The classical Kirwan blow-up $\pi\colon \widehat{X}_{\mathrm{cl}} \to X_{\mathrm{cl}}$ is the open substack of semistable points $\left(X_{\mathrm{cl}}^{\mathrm{intr}}\right)^{\mathrm{ss}}$ of the intrinsic blow-up $X_{\mathrm{cl}}^{\mathrm{intr}}$ whose complement is the unstable locus

$$\overline{(\pi^{\mathrm{intr}})^{-1}\left(q^{-1}(M^{\max}) \setminus X_{\mathrm{cl}}^{\max}\right)} \subseteq X_{\mathrm{cl}}^{\mathrm{intr}}.$$

An important property established in [Sav20] is the existence of a good moduli space $\widehat{X}_{\mathrm{cl}} \to \widehat{M}$ for the Kirwan blow-up such that the natural induced morphism $\widehat{M} \to M$ is proper. By construction, the maximum stabilizer dimension of points in $\widehat{X}_{\mathrm{cl}}$ is strictly lower than that of X_{cl}.

7.2. **Derived intrinsic and Kirwan blow-ups.** If X is a derived Artin stack satisfying the assumptions of this section, we have constructed earlier a canonical, closed immersion $X^{\max}$ of X, which parametrizes points in X with stabilizer of maximum dimension.

As in the classical case, we may thus give the following definition.

Definition 7.1. The *derived intrinsic blow-up* of X is defined as the stack over X

$$\pi^{\mathrm{intr}}\colon X^{\mathrm{intr}} := \mathrm{Bl}_{X^{\max}} X \longrightarrow X.$$

As expected, the derived intrinsic blow-up is a canonical derived enhancement of the classical intrinsic blow-up.

Proposition 7.2. *The classical truncation of X^{intr} is the classical intrinsic blow-up $X_{\mathrm{cl}}^{\mathrm{intr}}$ of the classical truncation X_{cl} of X.*

Proof. By the base change property of $X^{\max}$ along a cover by affine, strongly étale morphisms from affine quotient stacks, we may reduce to the case of a derived affine quotient stack $[U/G]$ for which $[U/G]^{\max} = [U^{G^0}/G]$. But then the statement follows immediately by Theorem 6.3 by working with the connected component G^0 and G-equivariance. □

Mimicking the classical case, it is straightforward to define the derived Kirwan blow-up.

Definition 7.3. The *derived Kirwan blow-up* $\widehat{X}$ of X is defined as the open substack of X^{intr} whose classical truncation is the classical Kirwan blow-up $\widehat{X}_{\mathrm{cl}} \subseteq X^{\mathrm{intr}}_{\mathrm{cl}}$.

We summarize the fundamental properties of the derived intrinsic and Kirwan blow-up in the following theorem, which combines the above with the properties of the corresponding classical blow-ups given in [Sav20, Theorem-Construction 4.7].

Theorem 7.4. *There exists a canonical derived Artin stack $\widehat{X}$, called the derived Kirwan blowup of X, together with a morphism $\pi \colon \widehat{X} \to X$, such that:*

(i) *Its classical truncation $\widehat{X}_{\mathrm{cl}}$ is of finite type over $\mathbb{C}$, has affine diagonal and admits a good moduli space morphism $\widehat{q} \colon \widehat{X}_{\mathrm{cl}} \to \widehat{M}$ with affine diagonal.*

(ii) *The maximum stabilizer dimension of closed points in $\widehat{X}$ is strictly smaller than that of X.*

(iii) *For any affine, strongly étale morphism $[U/G] \to X$ with U affine, the base change $\widehat{X} \times_X [U/G]$ is naturally isomorphic to the derived Kirwan blowup of $[U/G]$.*

(iv) *$\pi|_{\pi^{-1}(X^{\mathrm{s}})}$ is an isomorphism over the open locus X^{s} of properly stable points, i.e closed points $x \in X$ with finite stabilizer such that $q^{-1}(q(x)) = \{x\}$.*

$\widehat{X}$ is the semistable locus $(X^{\mathrm{intr}})^{\mathrm{ss}} \subseteq X^{\mathrm{intr}}$, an open substack of the Artin stack $X^{\mathrm{intr}} = \mathrm{Bl}_{X^{\max}} X$, called the intrinsic blowup of X.

The classical truncations of X^{intr} and $\widehat{X}$ are the classical intrinsic and Kirwan blow-ups of the classical truncation X_{cl} respectively.

7.3. **Derived stabilizer reduction of Artin stacks.** We are now in position to carry out the derived stabilizer reduction procedure, as described in the introductory part of the section.

By Theorem 7.4, the Kirwan blow-up $\widehat{X}$ preserves the assumed properties of X. In particular, we may repeatedly apply it until the maximum stabilizer dimension becomes zero and we obtain a derived Deligne–Mumford stack $\widetilde{X} := X^m$ by the canonical sequence

$$X^0 = X,\ X^1 := \widehat{X}^0,\ \ldots,\ X^m := \widehat{X}^{m-1}. \tag{7.4}$$

Definition 7.5. The derived Deligne–Mumford stack $\widetilde{X} \to X$ is called the derived stabilizer reduction of X.

It is clear by Theorem 7.4 that the classical truncation of $\widetilde{X}$ is the intrinsic stabilizer reduction of X_{cl}.

Remark 7.6. By Remark 6.5, if X is a classical Artin stack of finite type, then its derived stabilizer reduction $\widetilde{X}$ is well-defined, with classical truncation the intrinsic stabilizer reduction of X_{cl}.

7.4. **Connections with other approaches.** We conclude with a brief discussion regarding the connections between the following stabilizer reduction approaches in the literature:

(i) Kirwan's original desingularization procedure for classical, smooth quotient stacks obtained by Geometric Invariant Theory (GIT) [Kir85].
(ii) Edidin–Rydh's canonical reduction of stabilizers for classical (possibly singular) Artin stacks with good moduli spaces [ER21].
(iii) The intrinsic stabilizer reduction procedure introduced in [KLS17, Sav20] for (possibly singular) GIT quotient stacks and Artin stacks with good moduli spaces.
(iv) The derived stabilizer reduction procedure of the present paper.

In general, (ii) is (in spirit) a generalization of (i), while we have shown that (iv) is always a derived enhancement of (iii).

Both (ii) and (iii) are sequences of two-step operations consisting of a blow-up and the deletion of certain unstable points (this two-step operation in (ii) is called a saturated blow-up, while in (iii) a Kirwan blow-up). The main difference between (ii) and (iii) is in the blow-up being used: In (ii), it is the classical blow-up $\mathrm{Bl}_{X_{\mathrm{cl}}^{\max}} X_{\mathrm{cl}}$ of a classical Artin stack X_{cl} along the classical locus $X_{\mathrm{cl}}^{\max}$, while in (iv) it is the derived blow-up $\mathrm{Bl}_{X_{\mathrm{cl}}^{\max}} X_{\mathrm{cl}}$ of X_{cl} viewed as a derived stack along the derived locus $X_{\mathrm{cl}}^{\max}$.

For classical GIT quotient stacks that are smooth, all four constructions coincide: approaches (i)-(iii) are literally identical, while in (iv) it suffices to notice that the derived intrinsic blow-up of a classical smooth Artin stack is the same as the classical intrinsic blow-up (since the derived $X^{\max}$ is smooth and coincides with the classical $X^{\max}$) so that the derived Kirwan blow-up is classical and equals the saturated blow-up in each step.

However, in the singular case, the derived $X_{\mathrm{cl}}^{\max}$ is in general not isomorphic, but a derived enhancement of the classical $X_{\mathrm{cl}}^{\max}$, as can be seen in the following example. Hence, the outputs of (ii) and (iv) will typically differ.

Example 7.7. Suppose that $G = \mathbb{C}^\times$ is the one-dimensional torus acting on the affine plane $V = \mathbb{C}^2_{x,y}$ with weights 1 and -1 on the coordinates x and y respectively and $U \subseteq V$ is the closed G-invariant subscheme cut out by the ideal $I = (xy)$. Let $X = [U/G]$.

Then the classical $X^{\max}$ is the quotient stack $\operatorname{Spec} \mathbb{C}[x,y]/(x,y) \times BG$. To describe the derived $X^{\max}$, we can find a $B \in \mathsf{Alg}^G$ that is equivalent to $\mathcal{O}_U$ and then apply Proposition 4.26. In this case, it is clear that we can take $B(0) = \mathbb{C}[x,y]$ and $M_0 = B(0) \otimes_{\mathbb{C}} W_0$ where W_0 is the trivial 1-dimensional G-representation and $M_0 \to B(0)$ is multiplication by xy. But then, B_G is freely generated by $\mathbb{C}[x,y]/(x,y)$ in degree 0 and W_0 in degree -1, therefore the derived $X^{\max} \simeq \operatorname{Spec} B_G \times BG$ is not classical and differs from its classical counterpart.

8. The case of (-1)-shifted symplectic stacks

(-1)-shifted symplectic derived Artin stacks play a fundamental role in enumerative geometry and Donaldson–Thomas (DT) theory, since they naturally arise as derived moduli stacks of sheaves and perfect complexes on Calabi–Yau threefolds. In [KLS17, Sav20], it is shown that if X is such a stack, then the intrinsic stabilizer reduction $\widetilde{X}_{\mathrm{cl}}$ admits a semi-perfect obstruction theory of virtual dimension zero and hence by [CL11] a virtual fundamental cycle $[\widetilde{X}_{\mathrm{cl}}^{\mathrm{vir}}] \in A_0(\widetilde{X}_{\mathrm{cl}})$, whose degree was defined to be the generalized DT invariant via Kirwan blow-ups associated to X_{cl}. This was possible by rather cumbersome local computations.

In this section, we pursue a more detailed study of the derived stabilizer reduction $\widetilde{X}$ to put this result on a much more robust footing. Namely, we will show that by truncating the derived tangent complex of $\widetilde{X}$ there is a natural way to recover the above semi-perfect obstruction theory. As an application, we construct generalized Vafa–Witten invariants in the next section.

8.1. **Brief review of shifted symplectic geometry.** Shifted symplectic structures were introduced by Pantev–Toën–Vaquié–Vezzosi in [PTVV13]. Their definition is given in the affine case first, and then generalized by showing that the local notion satisfies smooth descent.

The local definition is as follows: For $X = \operatorname{Spec} A$ and all $p \geq 0$ consider the exterior power complex $(\Lambda^p \mathbb{L}_X, \delta) \in \mathsf{Mod}_A$, where the differential δ is induced by the differential of the algebra A. For a fixed $k \in \mathbb{Z}$, define a k-shifted p-form on X to be an element $\omega^0 \in (\Lambda^p \mathbb{L}_X)^k$ such that $\delta\omega^0 = 0$. To define closedness, consider the de Rham differential $\mathrm{d}\colon \Lambda^p \mathbb{L}_X \to \Lambda^{p+1}\mathbb{L}_X$. A *$k$-shifted closed p-form* is a sequence $(\omega^0, \omega^1, \ldots)$, with $\omega^i \in (\Lambda^{p+i}\mathbb{L}_X)^{k-i}$, such that $\delta\omega^0 = 0$ and $\mathrm{d}\omega^i + \delta\omega^{i+1} = 0$. When $p = 2$, any k-shifted 2-form $\omega^0 \in (\Lambda^2\mathbb{L}_X)^k$ induces a morphism $\omega^0 \colon \mathbb{T}_X \to \mathbb{L}_X[k]$, and we say that ω^0 is *non-degenerate* if this morphism is an equivalence.

Definition 8.1 ([PTVV13, Definition 1.18])**.** A k-shifted closed 2-form $\omega = (\omega^0, \omega^1, \ldots)$ is called a *k-shifted symplectic structure* if ω^0 is non-degenerate. We say that (X, ω) is a k-shifted symplectic (affine) derived scheme.

8.2. **Local structure of (-1)-shifted symplectic stacks.** We first introduce notation that will help us denote cdga's more efficiently.

Definition 8.2. Let G be a reductive group, V a smooth G-scheme, $\mathcal{W}^{-1}$ and $\mathcal{W}^{-2}$ be G-equivariant vector bundles on V and $\delta^{-1}\colon \mathcal{W}^{-1} \to \mathcal{O}_V$, $\delta^{-2}\colon \mathcal{W}^{-2} \to \mathcal{W}^{-1}$ be two G-equivariant morphisms.

Then we write $K(G, V, \mathcal{W}^{-1}, \mathcal{W}^{-2}, \delta^{-1}, \delta^{-2})$ for the G-equivariant sheaf of cdga's generated by $\mathcal{O}_V, \mathcal{W}^{-1}$ and $\mathcal{W}^{-2}$ in degrees $0, -1$ and -2 respectively and whose differential is determined by the maps δ^{-1}, δ^{-2}.

For brevity we write $\mathcal{W}_i$ for $(\mathcal{W}^{-i})^\vee$, where $i = 1, 2$, and δ_2 for $(\delta^{-2})^\vee$.

In the case where $\mathcal{W}^{-i} = W^{-i} \otimes_{\mathbb{C}} \mathcal{O}_V$ for G-representations W^{-i}, we also use the shorthand notation $K(G, V, W^{-1}, W^{-2}, \delta^{-1}, \delta^{-2})$.

Proposition 8.3. *Let $A = K(G, V, \mathcal{W}^{-1}, \mathcal{W}^{-2}, \delta^{-1}, \delta^{-2})$ where V is affine and write $U = \operatorname{Spec} A$. Then, after possible (Zariski) equivariant shrinking* *around fixed points $x \in U^G$, A can be written in the form*

$$A = K(G, V, W^{-1}, W^{-2}, \delta^{-1}, \delta^{-2})$$

for some G-representations W^{-1}, W^{-2}.

Proof. Lemma 4.25 applies verbatim to prove the statement. $\square$

The following lemma is a strengthening of the preceding proposition in the (-1)-shifted symplectic case.

Lemma 8.4. *Let $U = \operatorname{Spec} A$ and $X = [U/G]$ be a (-1)-shifted symplectic derived quotient stack with $x \in U^G$. Then after possible (Zariski) equivariant shrinking around x, we may assume that $A = K(G, V, W^{-1}, W^{-2}, \delta^{-1}, \delta^{-2})$, where:*

(i) *W^{-1} is the vector space spanned by $\frac{\partial}{\partial x_i}$ for a set of étale coordinates $\{x_i\}$ of V on which G acts linearly, so that $W^{-1} \otimes_{\mathbb{C}} \mathcal{O}_V \cong T_V$ as G-equivariant vector bundles on V.*

(ii) *Under this identification, δ^{-1} maps each $\frac{\partial}{\partial x_i}$ to $\frac{\partial f}{\partial x_i} \in \mathcal{O}_V$ for a G-invariant regular function $f \colon V \to \mathbb{A}^1$.*

(iii) *$W^{-2} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_V$ and the map $\delta^{-2} \colon \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_V = W^{-2} \to W^{-1} \otimes_{\mathbb{C}} \mathcal{O}_V \cong T_V$ is the infinitesimal derivative of the G-action on V.*

Proof. By Lemma 4.25, we may assume that A is a G-equivariant cdga of standard form, i.e. generated by $\mathcal{O}_V$ in degree 0, where V is a smooth affine G-scheme, and G-equivariant vector bundles $\mathcal{W}^{-i}$ on V in negative degrees. Write as usual $W^{-i} = \mathcal{W}^{-i}|_x$. We then have

$$\mathbb{L}_X|_x = \left[\ldots \longrightarrow \mathcal{W}^{-2}|_x \longrightarrow \mathcal{W}^{-1}|_x \longrightarrow \Omega_V|_x \longrightarrow \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{O}_V \right].$$

Moreover, we may assume that $\operatorname{Spec} A$ is minimal at x, meaning that the differentials of $\mathbb{L}_X|_x$ vanish. Since X is (-1)-shifted symplectic, there exists a quasi-isomorphism $\mathbb{L}_X|_x \cong \mathbb{L}_X|_x^\vee[1]$ and then minimality implies that we must have $W^{-i} = 0$ for $i > 2$. Moreover, we get G-equivariant isomorphisms $W^{-1} \cong T_V|_x, W^{-2} \cong \mathfrak{g}$ and we can pick a set of étale coordinates on V with a linear G-action using a basis for $\Omega_V|_x$ and lifting through $\mathrm{d} \colon \mathcal{O}_V \to \Omega_V$.

Consider now the smooth morphism $\operatorname{Spec} A \to [\operatorname{Spec} A/G]$. The pullback of ω to $\operatorname{Spec} A$ is a G-invariant (-1)-shifted closed 2-form. Since G is reductive, the arguments of [BBJ19, Subsection 5.2] and the proof of [BBJ19, Theorem 5.18] extend verbatim in this G-equivariant setting to show that the differential δ^{-1} maps $\frac{\partial}{\partial x_i}$ to $\frac{\partial f}{\partial x_i} \in \mathcal{O}_V$ for a G-invariant regular function $f \in \mathcal{O}_V^G$.

Finally, we may restrict the quasi-isomorphism $\mathbb{L}_X \cong \mathbb{L}_X^\vee[1]$ to U along the G-equivariant embedding $U \to V$ to obtain a G-equivariant morphism of complexes

$$\begin{tikzcd}
\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_U \arrow[r, "d\delta^{-2}"] \arrow[d] & T_V|_U \arrow[r, "d\delta^{-1}"] \arrow[d, "\mathrm{id}"] & \Omega_V|_U \arrow[r, "\sigma"] \arrow[d, "\mathrm{id}"] & \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{O}_U \arrow[d] \\
\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_U \arrow[r, "\sigma^\vee"] & T_V|_U \arrow[r, "(d\delta^{-1})^\vee"] & \Omega_V|_U \arrow[r, "(d\delta^{-2})^\vee"] & \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{O}_U,
\end{tikzcd}$$

where the two middle vertical arrows are the identity maps and, by symmetry, $(d\delta^{-1})^\vee = d\delta^{-1}$.

The differentials of both complexes vanish at x and hence it follows that the two outer vertical arrows must be isomorphisms after possible shrinking around a G-invariant Zariski neighbourhood of x. Thus we may identify $d\delta^{-2}$ with the infinitesimal derivative $\sigma^\vee$ of the G-action. By comparing degrees, $d\delta^{-2}$ determines the morphism $\delta^{-2}\colon \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_V \to T_V$, which thus must equal $\sigma^\vee$, so we have established (i)-(iii). $\square$

Remark 8.5. In fact, we can prove a slightly stronger statement giving an equivariant version of the derived Darboux theorem of [BBBJ15]. Namely, using the description of shifted symplectic structures on quotient stacks $[\operatorname{Spec} A/G]$ [Yeu21], the (-1)-shifted symplectic form on X is can be brought to a canonical Darboux form.

Remark 8.6. We will actually only need conditions (i) and (iii) in the ensuing computations in this paper.

8.3. **An intrinsic blow-up computation.** In order to work with the tangent complex of the derived stabilizer reduction of a (-1)-shifted symplectic stack later, we will need to get a better handle on the local structure of its iterated intrinsic blow-ups. To this end, we introduce further terminology.

Definition 8.7. Let $A = K(G, V, \mathcal{W}^{-1}, \mathcal{W}^{-2}, \delta^{-1}, \delta^{-2})$.

We say that A satisfies property (†) if there exists a G-invariant Cartier divisor D on V and a commutative diagram of G-equivariant morphisms

$$\begin{tikzcd}
\Omega_V(-D) \arrow[r] \arrow[dr] & \mathcal{W}_1 \arrow[d] \arrow[dr, "\delta_2"] & \\
& \mathfrak{g}^\vee(-D) \arrow[r] \arrow[dr] & \mathcal{W}_2 \arrow[d] \\
& & \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{O}_V
\end{tikzcd} \tag{8.1}$$

where $\Omega_V(-D) \to \mathfrak{g}^\vee(-D)$ is induced by the derivative of the G-action, $\mathfrak{g}^\vee(-D) \to \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{O}_V$ is the natural inclusion, $\mathcal{W}_2 \to \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{O}_V$ is injective, and for any point in V with closed G-orbit and reductive stabilizer $H \subseteq G$, the composition

$$\mathcal{W}_1|_{V^H} \to \mathfrak{g}^\vee(-D) \to \mathfrak{h}^\vee(-D)$$

vanishes.

Proposition 8.8. *Suppose that A satisfies property (†) and $X = [\operatorname{Spec} A/G]$ is Deligne–Mumford. The truncation $E_\bullet = \tau^{[0,1]}\mathbb{T}_X|_{X_{\mathrm{cl}}}$ is a perfect two-term complex, whose dual $E^\bullet$ defines a perfect obstruction theory on X_{cl}.*

Proof. By assumption, the injection $\mathfrak{g}^\vee(-D) \to \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{O}_V$ factors through the injection $\mathcal{W}_2 \to \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{O}_V$ and hence $\mathfrak{g}^\vee(-D) \to \mathcal{W}_2$ is injective. In particular, it follows that the kernel of δ_2 is equal to the kernel of the morphism $\mathcal{W}_1 \to \mathfrak{g}^\vee(-D)$. Since X is Deligne–Mumford, the dual of the derivative $\Omega_V(-D) \to \mathfrak{g}^\vee(-D)$ is surjective, and thus so is $\mathcal{W}_1 \to \mathfrak{g}^\vee(-D)$. Therefore, the kernel $\mathcal{F}_1 = \ker \delta_2$ is a vector bundle.

We then have (suppressing the restriction to X_{cl} on the right-hand side)

$$\mathbb{T}_X|_{X_{\mathrm{cl}}} = [\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_V \longrightarrow T_V \xrightarrow{(d\delta^{-1})^\vee} \mathcal{W}_1 \xrightarrow{\delta_2} \mathcal{W}_2]$$

and, writing $T_{V/G}$ for the cokernel of the derivative $\mathfrak{g} \to T_V$,

$$E_\bullet = \tau^{[0,1]}\mathbb{T}_X|_{X_{\mathrm{cl}}} = [T_{V/G} \longrightarrow \mathcal{F}_1],$$

a two-term complex of vector bundles.

To see that $E^\bullet = (E_\bullet)^\vee$ defines a perfect obstruction theory, let $I = \mathrm{im}(\delta^{-1})$ be the ideal sheaf of X_{cl} inside V. Since δ_1 factors through $\mathcal{F}_1$, we have a commutative diagram

$$\begin{tikzcd}
{[\mathcal{W}^{-2}} \arrow[r, "\delta^{-2}"] & \mathcal{W}^{-1} \arrow[r, "d\delta^{-1}"] \arrow[d] & \Omega_V \arrow[r] \arrow[d, equal] & \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{O}_V] \arrow[r, equal] \arrow[d, equal] & \mathbb{L}_X|_{X_{\mathrm{cl}}} \arrow[dd] \\
 & {[\mathcal{F}_1^\vee} \arrow[r] \arrow[d] & \Omega_V \arrow[r] \arrow[d, equal] & \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{O}_V] \arrow[d, equal] & \\
 & {[I/I^2} \arrow[r, "d"] & \Omega_V \arrow[r] & \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{O}_V] \arrow[r, equal] & \mathbb{L}_{X_{\mathrm{cl}}}^{\geq -1},
\end{tikzcd}$$

where both left-most vertical arrows are surjective.

It follows that $E^\bullet = [\mathcal{F}_1^\vee \to \Omega_{V/G}]$ gives a perfect obstruction theory on X_{cl}, where $\Omega_{V/G} = T_{V/G}^\vee = \ker(\Omega_V \to \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{O}_V)$. $\square$

Let now $X = \operatorname{Spec} A$, where $A = K(G, V, W^{-1}, W^{-2}, \delta^{-1}, \delta^{-2})$ satisfies property (†) and $x \in X$ be a point with closed G-orbit and (reductive) stabilizer H of maximal dimension.

Using Luna's étale slice theorem [Dré04], we take a slice $x \in S \subseteq V$ for the G-action on V at x giving rise to an affine, strongly étale morphism $\phi\colon S \times_H G \to V$.

The isomorphism $[S/H] \cong [S \times_H G/G]$ then gives an equivalence between H-equivariant sheaves on S and G-equivariant sheaves on $S \times_H G$. We will freely use this identification henceforth.

The following proposition follows from the definitions.

Proposition 8.9. *Let $B = K(G, S \times_H G, W^{-1}, W^{-2}, \phi^*\delta^{-1}, \phi^*\delta^{-2})$ and set $U = \operatorname{Spec} B$. Then B also satisfies property (†) and the natural morphism $U \to X$ is G-equivariant with $[U/G] \to [X/G]$ strongly étale.*

Moreover, if we write $(W^{-i})^{\mathrm{fix},H}$ for the fixed part of W^{-i} considered as an H-representation, $(\mathcal{W}^{-i})^{\mathrm{fix},H}$ for the corresponding G-equivariant summand of $\mathcal{W}^{-i}$ under the isomorphism $[S/H] \cong [S \times_H G/G]$, and let

$$C = K\left(G, S^H \times_H G, (\mathcal{W}^{-1})^{\mathrm{fix},H}, (\mathcal{W}^{-2})^{\mathrm{fix},H}, \delta^{-1}|_{S^H \times_H G}, \delta^{-2}|_{S^H \times_H G}\right),$$

then $[\operatorname{Spec} C/G]$ is $[U/G]^{\max}$.

We are now able to explicitly compute the blow-up of U along $U^G = \operatorname{Spec} C$, after setting up some more notation.

Without loss of generality, we may assume that $x_1, \ldots, x_l$ are the H-moving coordinates of S, which lift to minimal generators of the ideal of $S^H \times_H G$ in $S \times_H G$ and on which the H-action is linear.

Pick bases $\{\underline{w}^{-1,f}\}$ for $(W^{-1})^{\mathrm{fix},H}$ and $\{\underline{w}^{-1,m}\}$ for $(W^{-1})^{\mathrm{mv},H}$. Denote their union by $\{\underline{w}^{-1}\}$.

By H-equivariance, the images of $\{\underline{w}^{-1,m}\}$ in $S \times_H G$ under δ^{-1} must land in the ideal $(x_1, \dots, x_l)$. Write $s_i^{\mathrm{mv}} = \sum_k \alpha_{ik} x_k$ for the image of the i-th basis element $w_i^{-1,m}$ and $s_i^{\mathrm{fix}} \in \mathcal{O}_V$ for the image of the i-th basis element $w_i^{-1,f}$.

Since $\delta^{-2}|_{V^H}$ vanishes (using property (†)), picking a basis for $(W^{-2})^{\mathrm{mv},H}$, the images of its elements in $\mathcal{W}^{-1}$ must land inside $(x_1, \dots, x_l)\mathcal{W}^{-1}$ by G-equivariance. Write $p_j^{\mathrm{mv}} := \sum_{k,\ell} \beta_{jk\ell} x_\ell \cdot w_k^{-1}$ for the image of the j-th basis element. Picking a basis for $(W^{-2})^{\mathrm{fix},H}$, write p_j^{fix} for the image of the j-th basis element $w_j^{-2,\mathrm{fix}}$.

Lemma 8.10. *Let $U = \operatorname{Spec} B$ and $U^G = \operatorname{Spec} C$ be as in the previous proposition. Then*

$$R_{U^G/U} = \frac{B[t^{-1}, \underline{v}]}{(\underline{v}t^{-1} - \underline{x}, \{\sum_k \alpha_{ik} v_k\}, \{\sum_{k,\ell} \beta_{jk\ell} v_\ell \cdot w_k^{-1}\})}. \tag{8.2}$$

Proof. For brevity, denote (by slight abuse of notation) $V = S \times_H G$. Define:

(i) Q to be the derived affine G-scheme with

$$\mathcal{O}_Q = \mathcal{O}_U/(\underline{x}). \tag{8.3}$$

(ii) Y to be the derived affine G-scheme with

$$\mathcal{O}_Y = \mathcal{O}_V/(\underline{s}^f) \tag{8.4}$$

obtained by the G-equivariant cell attachment $(W^{-1})^{\mathrm{fix},H}[0] \to \mathcal{O}_V$ in degree 0. Observe that we can also write

$$\mathcal{O}_Q = \mathcal{O}_Y/(\underline{x}, \underline{s}^{\mathrm{mv}}, \underline{0}), \tag{8.5}$$

with the appropriate G-equivariant cell attachments corresponding to $\underline{x}, \underline{s}^{\mathrm{mv}}$ understood and the cell attachment $W^{-2}[1] \to \mathcal{O}_Y$ given by the zero morphism. This is possible due to the fact that the natural morphism $W^{-2}[1] \to \mathcal{O}_U$ restricts to zero after quotienting out by the ideal $(\underline{x})$ by assumption.

By definition, we similarly have that

$$\mathcal{O}_{U^G} = \mathcal{O}_Y/(\underline{x}, \underline{0}^{\mathrm{fix}}), \tag{8.6}$$

where the cell attachment for $\underline{x}$ is as above and we use the zero cell attachment $(W^{-2})^{\mathrm{fix},H}[1] \to \mathcal{O}_Y$ for $\underline{0}^{\mathrm{fix}}$.

We have an induced commutative square of closed embeddings

$$\begin{tikzcd} Z = U^G \arrow[r] \arrow[d] & U \arrow[d] \\ Q \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

and hence by Lemma 3.12 it follows that

$$R_{U^G/U} = R_{Q/U} \otimes_{R_{Q/Y}} R_{Z/Y}.$$

Using (8.3), (8.4), (8.5), (8.6) and the (G-equivariant) finite quotient formula of Proposition 4.29, we obtain

$$R_{Q/U} = \frac{\mathcal{O}_U[t^{-1}, \underline{v}]}{(\underline{v}t^{-1} - \underline{x})}$$

$$R_{Z/Y} = \frac{\mathcal{O}_Y[t^{-1}, \underline{v}, \underline{u}^{\mathrm{fix}}]}{(\underline{v}t^{-1} - \underline{x},\ \underline{u}^{\mathrm{fix}}t^{-1})}$$

$$R_{Q/Y} = \frac{\mathcal{O}_Y[t^{-1}, \underline{v},\ \underline{e}^{\mathrm{mv}}, \underline{u}]}{(\underline{v}t^{-1} - \underline{x},\ \underline{e}^{\mathrm{mv}}t^{-1} - \underline{s}^{\mathrm{mv}},\ \underline{u}t^{-1})}.$$

The morphism $R_{Q/Y} \to R_{Z/Y}$ maps $\underline{e}^{\mathrm{mv}}$ and $\underline{u}^{\mathrm{mv}}$ to zero, while the morphism $R_{Q/Y} \to R_{Q/U}$ maps $\underline{e}_i^{\mathrm{mv}}$ to $\sum_k \alpha_{ik} v_k \in R_{Q/U}$ and u_j^{mv} to $\sum_{k,\ell} \beta_{jk\ell} v_\ell \cdot w_k^{-1} \in \mathcal{R}_{Q/U}$.

Thus, the homotopy pushout diagram

$$\begin{tikzcd}
\mathbb{Z}[\underline{e}^{\mathrm{mv}}, \underline{u}^{\mathrm{mv}}] \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
R_{Q/Y} \arrow[r] \arrow[d] & R_{Z/Y} \arrow[d] \\
R_{Q/U} \arrow[r] & R_{U^G/U}
\end{tikzcd}$$

immediately gives (8.2). □

We obtain the following immediate corollary.

Corollary 8.11. *In the above notation, write $Y = S \times_H G$ and $GY^H = S^H \times_H G$. Then $Y^{\mathrm{intr}} = \mathrm{Bl}_{GY^H} Y$ and $\pi\colon \widehat{Y} = (\mathrm{Bl}_{GY^H} Y)^{\mathrm{ss}} \to Y$ with exceptional divisor E.*

Consider the sheaf of cdga's $\widehat{B}$ generated by $\mathcal{O}_{\widehat{Y}}$ in degree 0, $\pi^*(\mathcal{W}^{-1})^{\mathrm{fix},H} \oplus \pi^*(\mathcal{W}^{-1})^{\mathrm{mv},H}(E)$ in degree -1 and $\pi^*(\mathcal{W}^{-2})^{\mathrm{fix},H} \oplus \pi^*(\mathcal{W}^{-2})^{\mathrm{mv},H}(2E)$ in degree -2 with differentials induced by (8.2).

Then $\widehat{B}$ satisfies property (†) and the derived Kirwan blow-up $\widehat{U}$ is given by $\mathrm{Spec}\,\widehat{B}$.

Proof. We only need to check that property (†) is satisfied. But this follows verbatim from the argument used in the proof of [KLS17, Lemma 5.3]. □

8.4. **Perfectness of the truncated tangent complex, virtual fundamental cycle and virtual structure sheaf.** Let now X be a derived Artin stack satisfying our usual assumptions, which include the existence of a good moduli space for the classical truncation X_{cl}.

Lemma 8.12. *Suppose that X admits a strongly étale cover by quotient stacks* $[\mathrm{Spec}\,A/G]$ *where A satisfies property* (†)*. Then the same holds for its Kirwan blow-up $\widehat{X}$.*

Proof. This is immediate from Corollary 8.11. □

Corollary 8.13. *The derived stabilizer reduction $\widetilde{X}$ of a (-1)-shifted symplectic derived Artin stack X admits a strongly étale cover by quotient stacks* $[\mathrm{Spec}\,A/G]$ *where A satisfies property* (†).

Proof. By Proposition 5.10, we know that X admits a strongly étale cover by affine quotient stacks $[\operatorname{Spec} A/G]$. Using Lemma 8.4, we deduce that after possible equivariant shrinking, A may be taken to satisfy property (†). Using the Fundamental Lemma [Alp10, Theorem 6.10], further shrinking ensures that the morphism $[\operatorname{Spec} A/G] \to X$ remains strongly étale.

Since $\widetilde{X}$ is obtained by X through a sequence of Kirwan blow-ups, the claim follows by Lemma 8.12 and induction. □

This corollary together with Proposition 8.8 imply the main result of this section. We use cohomological grading here.

Theorem 8.14. *Let X be a derived (-1)-shifted symplectic Artin stack of finite type and $\widetilde{X}$ its derived stabilizer reduction. Then the truncation $E_\bullet = \tau^{[0,1]}\mathbb{T}_{\widetilde{X}}|_{\widetilde{X}_{\mathrm{cl}}}$ is a perfect complex of virtual dimension* 0.

Moreover, let $\{U_\alpha \to \widetilde{X}\}$ be a strongly étale cover by quotient stacks $[\operatorname{Spec} A/G]$ where A satisfies property (†). Write $E^\bullet = (E_\bullet)^\vee$.

Then the complexes $E^\bullet|_{(U_\alpha)_{\mathrm{cl}}}$ are part of the data of a semi-perfect obstruction theory on $\widetilde{X}_{\mathrm{cl}}$, which recovers the semi-perfect obstruction theory on $\widetilde{X}_{\mathrm{cl}}$ constructed in [Sav20]*, and also of an almost perfect obstruction theory which recovers the one constructed in* [KS22].

In particular, $E_\bullet$ together with $\widetilde{X}$ recover the 0*-dimensional virtual fundamental cycle $[\widetilde{X}_{\mathrm{cl}}]^{\mathrm{vir}} \in A_0(\widetilde{X}_{\mathrm{cl}})$ and virtual structure sheaf $[\mathcal{O}^{\mathrm{vir}}_{\widetilde{X}_{\mathrm{cl}}}] \in K_0(\widetilde{X}_{\mathrm{cl}})$ of the intrinsic stabilizer reduction $\widetilde{X}_{\mathrm{cl}}$, constructed in* [Sav20, KS22].

Proof. The statement on the perfectness and virtual dimension of $E_\bullet$ follows immediately by Proposition 8.8 and the preceding corollary.

Now, by Proposition 8.8, the complexes $E^\bullet|_{(U_\alpha)_{\mathrm{cl}}}$ define perfect obstruction theories $\phi_\alpha \colon E^\bullet|_{(U_\alpha)_{\mathrm{cl}}} \to \mathbb{L}^{\geq -1}_{(U_\alpha)_{\mathrm{cl}}}$ on $(U_\alpha)_{\mathrm{cl}}$. We may assume that the U_α are affine schemes after étale base change, since U_α are Deligne–Mumford.

Write $\mathbb{T}^{\leq 1}_{\widetilde{X}_{\mathrm{cl}}} = (\mathbb{L}^{\geq -1}_{\widetilde{X}_{\mathrm{cl}}})^\vee$. Then, by construction of the ϕ_α, we have commutative diagrams

$$\begin{tikzcd}
\mathbb{T}^{\leq 1}_{\widetilde{X}_{\mathrm{cl}}}|_{(U_\alpha)_{\mathrm{cl}}} \arrow[r, "\phi_\alpha^\vee"] \arrow[dr, "\rho|_{(U_\alpha)_{\mathrm{cl}}}"'] & E_\bullet|_{(U_\alpha)_{\mathrm{cl}}} \arrow[d, "\psi|_{(U_\alpha)_{\mathrm{cl}}}"] \\
 & \mathbb{T}_{\widetilde{X}}|_{(U_\alpha)_{\mathrm{cl}}},
\end{tikzcd}$$

where $\psi \colon E_\bullet \to \mathbb{T}_{\widetilde{X}}|_{\widetilde{X}_{\mathrm{cl}}}$ is the natural morphism, and ρ is the dual of the composition $\mathbb{L}_{\widetilde{X}}|_{\widetilde{X}_{\mathrm{cl}}} \to \mathbb{L}_{\widetilde{X}_{\mathrm{cl}}} \to \mathbb{L}^{\geq -1}_{\widetilde{X}_{\mathrm{cl}}}$.

In particular, restricting to any closed point $p \in (U_\alpha)_{\mathrm{cl}}$ and taking $h^1(-)$, we obtain commutative diagrams

$$\begin{tikzcd}
h^1(\mathbb{T}^{\leq 1}_{\widetilde{X}_{\mathrm{cl}}}|_p) \arrow[r, "h^1(\phi_\alpha^\vee|_p)"] \arrow[dr, "h^1(\rho|_p)"'] & h^1(E_\bullet|_p) \arrow[d, "h^1(\psi|_p)"] \\
 & h^1(\mathbb{T}_{\widetilde{X}}|_p),
\end{tikzcd} \tag{8.7}$$

We have a global obstruction sheaf defined by $\mathrm{Ob}_{X_{\mathrm{cl}}} = h^1(E_\bullet)$, which glues together the local obstruction sheaves $\mathrm{Ob}_\alpha = h^1(E_\bullet|_{U_\alpha})$ and the transition functions are the same as the ones used in [Sav20], by construction.

It is now a simple verification to check that the local obstruction theories satisfy the necessary compatibility condition to define a semi-perfect obstruction theory as in [CL11] and that we hence get the same virtual cycle: Let $(U_{\alpha\beta})_{\mathrm{cl}} = (U_\alpha)_{\mathrm{cl}} \times_{\widetilde{X}_{\mathrm{cl}}} (U_\beta)_{\mathrm{cl}}$. In the terminology of [KLS17, Definition 4.2], consider

$$\begin{tikzcd} \Delta \arrow[r, "g"] \arrow[d, "\iota"'] & (U_{\alpha\beta})_{\mathrm{cl}} \arrow[d] \\ \bar{\Delta} \arrow[r] \arrow[ur, dashed, "\bar{g}"'] & \widetilde{X}_{\mathrm{cl}}, \end{tikzcd}$$

an infinitesimal lifting problem at a closed point $p \in (U_{\alpha\beta})_{\mathrm{cl}}$, where $\bar{\Delta}$ is an extension of Δ by an ideal I. By [CL11, Lemma 2.6], this induces a canonical obstruction element

$$\omega(g, \Delta, \bar{\Delta}) \in \mathrm{Ext}^1(g^*\mathbb{L}_{\widetilde{X}_{\mathrm{cl}}}, I) = h^1(\mathbb{T}^{\leq 1}_{\widetilde{X}_{\mathrm{cl}}}|_p) \otimes_{\mathbb{C}} I.$$

We need to show that

$$h^1(\phi_\alpha^\vee|_p)(\omega(g, \Delta, \bar{\Delta})) = h^1(\phi_\beta^\vee|_p)(\omega(g, \Delta, \bar{\Delta})) \in \mathrm{Ext}^1(g^*E^\bullet, I) = h^1(E_\bullet|_p) \otimes_{\mathbb{C}} I.$$

But this follows immediately from diagram (8.7), since $h^1(\psi|_p)$ is an injection by the definition of $E_\bullet$ and Proposition 8.8 and thus $h^1(\phi_\alpha^\vee|_p) = h^1(\phi_\beta^\vee|_p)$.

The claim for the almost perfect obstruction theory is similar and follows the reasoning of [KS22, Subsection 5.4] identically. We leave the details to the reader. □

9. Generalized Donaldson–Thomas and Vafa–Witten invariants

In this short section, we explore immediate applications of our results to enumerative geometry and Donaldson–Thomas theory in particular.

Namely, we obtain a fully derived upgrade of the construction of generalized Donaldson–Thomas invariants via Kirwan blow-ups developed in [KLS17, Sav20]. These act as virtual counts of semistable sheaves and perfect complexes on projective Calabi–Yau threefolds.

The robustness of derived geometry also allows us to apply the same method to define new intersection-theoretic generalized Vafa–Witten invariants, enumerating semistable Higgs pairs on projective surfaces.

Throughout, $T = \mathbb{C}^\times$ denotes the one-dimensional complex torus and W a smooth Calabi–Yau threefold (which will be clear from context).

9.1. **Generalized Donaldson–Thomas invariants via Kirwan blow-ups.** Let $\mathcal{M}$ be a derived moduli stack parametrizing semistable sheaves or perfect complexes on a smooth, projective Calabi–Yau threefold W, as considered in [Sav20]. Possible choices of stability here are Gieseker stability for sheaves and PT, polynomial or Bridgeland stability for complexes.

As in [Sav20], $\mathcal{M}$ is (-1)-shifted symplectic and its classical truncation $\mathcal{M}_{\mathrm{cl}}$ admits a good moduli space $\mathcal{M}_{\mathrm{cl}} \to M_{\mathrm{cl}}$ with M_{cl} proper.

Let $\mathcal{N}_{\mathrm{cl}} := \mathcal{M}_{\mathrm{cl}} /\!\!/ \mathbb{C}^\times$ be the rigidification of $\mathcal{M}_{\mathrm{cl}}$ by the scaling automorphisms of sheaves or complexes. In [Sav20], it is shown that the intrinsic stabilizer reduction $\widetilde{\mathcal{N}}_{\mathrm{cl}}$ of $\mathcal{N}$ admits a natural semi-perfect obstruction theory of virtual dimension zero. The degree of the induced virtual fundamental cycle $[\widetilde{\mathcal{N}}_{\mathrm{cl}}]^{\mathrm{vir}}$ is defined to be the generalized Donaldson–Thomas invariant via Kirwan blow-ups associated to $\mathcal{M}$. By [KS22], the obstruction theory can be upgraded to an almost perfect obstruction theory with induced virtual structure sheaf $[\mathcal{O}^{\mathrm{vir}}_{\widetilde{\mathcal{N}}_{\mathrm{cl}}}] \in K_0(\widetilde{\mathcal{N}}_{\mathrm{cl}})$.

Our results above allow us to obtain a canonical derived enhancement as follows.

Theorem 9.1. *There exists a derived Artin stack $\mathcal{N}$ with classical truncation $\mathcal{N}_{\mathrm{cl}}$. The truncation $E_\bullet = \tau^{[0,1]}\mathbb{T}_{\widetilde{\mathcal{N}}}|_{\widetilde{\mathcal{N}}_{\mathrm{cl}}}$ of the tangent complex is perfect, with induced virtual fundamental cycle (using Theorem 8.14) equal to $[\widetilde{\mathcal{N}}_{\mathrm{cl}}]^{\mathrm{vir}} \in A_0(\widetilde{\mathcal{N}}_{\mathrm{cl}})$ and induced virtual structure sheaf equal to $[\mathcal{O}^{\mathrm{vir}}_{\widetilde{\mathcal{N}}_{\mathrm{cl}}}] \in K_0(\widetilde{\mathcal{N}}_{\mathrm{cl}})$.*

Proof. The existence of $\mathcal{N}$ follows from [Hal20], which generalizes the rigidication construction to derived Artin stacks.

Now, observe that $\mathcal{M}$ admits a strongly étale cover by quotient stacks $[\operatorname{Spec} A/G]$ where A satisfies property (†). Since $\mathcal{N} = \mathcal{M} /\!\!/ \mathbb{C}^\times$, it follows that $\mathcal{N}$ admits a strongly étale cover by the quotient stacks $[\operatorname{Spec} A/(G/\mathbb{C}^\times)]$.

It is then clear that the arguments of Section 7 extend verbatim to show that Kirwan blow-ups preserve the form of these étale covers and, in addition, Deligne–Mumford stacks with such covers have perfect two-term truncation of their tangent complex in amplitude $[0,1]$, as desired.

The equality of virtual fundamental cycles and virtual structure sheaves follows identical arguments as for Theorem 8.14. □

9.2. **Stacks with a torus action.** Let $X \in \mathsf{Art}^T$ be a derived Artin stack with a T-action.

Proposition 9.2. *Let $X \in \mathsf{Art}^T$ be a stack whose T-action morphism is representable and separated. Then $X^{\max}$ admits a natural T-action and is a T-invariant closed substack of X.*

Proof. The claim follows from functoriality of $(-)^{\max}$ for representable and separated morphisms by Proposition 5.18(iii) and $(X \times T)^{\max} = X^{\max} \times T$. □

9.3. **Vafa–Witten invariants.** Let $(S, \mathcal{O}_S(1))$ be a polarized projective surface and $\pi\colon W = \operatorname{Tot}(K_S) \to S$ the total space of its canonical bundle with polarization $\pi^*\mathcal{O}_S(1)$.

By the results of [TT20, TT17], there exists a (-1)-shifted symplectic derived Artin stack $\mathcal{N}$, which parametrizes Gieseker semistable Higgs pairs $(E, \phi\colon E \to E \otimes K_S)$ on S with fixed positive rank and Chern classes, fixed determinant $\det E$ and zero trace $\operatorname{tr} \phi = 0$. $\mathcal{N}$ is a moduli stack of compactly supported Gieseker semistable sheaves on W with respect to the polarization $\pi^*\mathcal{O}_S(1)$. It admits a natural T-action induced by scaling the fibers of the projection map $W \to S$.

By construction, the classical truncation $\mathcal{N}_{\mathrm{cl}}$ is a GIT quotient stack $[Q^{\mathrm{ss}}/G]$, where Q is an appropriate Quot scheme on W and the T-action is induced by a T-action on Q^{ss}, which commutes with the G-action on Q^{ss}. Thus, the T-action is representable, and since $Q^{\mathrm{ss}} \times T$, Q^{ss} are separated, it is also separated, hence representable and separated. Moreover, the T-fixed locus $\mathcal{N}_{\mathrm{cl}}^T$ has a good moduli space N_{cl}^T which is a projective scheme.

We now see that, since $\mathcal{N}^T$ is T-invariant by Proposition 9.2, $\widetilde{\mathcal{N}}$ admits a T-action such that the projection $\widetilde{\mathcal{N}} \to \mathcal{N}$ is T-equivariant. Since N_{cl}^T is proper, it follows that $\widetilde{\mathcal{N}}_{\mathrm{cl}}$ is a proper Deligne–Mumford stack. Moreover, Lemma 8.12, Corollary 8.13 and Theorem 8.14 apply T-equivariantly and imply that $\tau^{[0,1]}\mathbb{T}_{\widetilde{\mathcal{N}}}|_{\widetilde{\mathcal{N}}_{\mathrm{cl}}}$ is a T-equivariant perfect complex on $\widetilde{\mathcal{N}}_{\mathrm{cl}}$ and the same is true for the T-fixed component of the restriction to the fixed locus $\widetilde{\mathcal{N}}_{\mathrm{cl}}^T$, i.e., $\tau^{[0,1]}\mathbb{T}_{\widetilde{\mathcal{N}}}|_{\widetilde{\mathcal{N}}_{\mathrm{cl}}^T}^{\mathrm{fix}}$, which we know is the same as $\tau^{[0,1]}\mathbb{T}_{\widetilde{\mathcal{N}}^T}|_{\widetilde{\mathcal{N}}_{\mathrm{cl}}^T}$. Write $N^{\mathrm{vir}} = \tau^{[0,1]}\mathbb{T}_{\widetilde{\mathcal{N}}}|_{\widetilde{\mathcal{N}}_{\mathrm{cl}}^T}^{\mathrm{mv}}$ for the moving component.

In particular, by [Kie18, KS20], the T-fixed locus $\widetilde{\mathcal{N}}_{\mathrm{cl}}^T$ admits a virtual fundamental cycle $[\widetilde{\mathcal{N}}_{\mathrm{cl}}^T]^{\mathrm{vir}}$ and also a virtual structure sheaf $[\mathcal{O}_{\widetilde{\mathcal{N}}_{\mathrm{cl}}^T}^{\mathrm{vir}}] \in K_0^T(\widetilde{\mathcal{N}}_{\mathrm{cl}}^T)$. Thus, mirroring the virtual torus localization formula [GP99, Lee04], we can give the following definition.

Definition 9.3. The numerical generalized Vafa–Witten invariant via Kirwan blow-ups associated to $\mathcal{N}$ is defined by the formula

$$\mathrm{VWK}(\mathcal{N}) = \int_{[\widetilde{\mathcal{N}}_{\mathrm{cl}}^T]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})}.$$

For any $\beta \in K_T^0(\widetilde{\mathcal{N}}_{\mathrm{cl}}^T) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Q}(t^{\frac{1}{2}})$, the β-twisted K-theoretic generalized Vafa–Witten invariant via Kirwan blow-ups associated to $\mathcal{N}$ is defined by the formula

$$\mathrm{VWK}^{\mathrm{K-th},\beta}(\mathcal{N}) = \chi_t\left(\widetilde{\mathcal{N}}_{\mathrm{cl}}^T, \frac{[\mathcal{O}_{\widetilde{\mathcal{N}}_{\mathrm{cl}}^T}^{\mathrm{vir}}]}{e(N^{\mathrm{vir}})} \otimes_{\mathcal{O}_{\widetilde{\mathcal{N}}_{\mathrm{cl}}^T}} \beta\right) \in \mathbb{Q}(t^{\frac{1}{2}}).$$

Here t denotes the torus parameter and χ_t has the same meaning as in [Tho18, Subsection 2.4].

It is interesting to investigate whether this definition satisfies the wall-crossing formulas of [TT17, Joy21]. Deformation invariance of $\mathrm{VWK}(\mathcal{N})$ follows from a T-equivariant version of the arguments in [Sav20].

Remark 9.4. The reason for the localization to $\mathbb{Q}(t^{\frac{1}{2}})$ above is to preserve consistency with the notation used in [Tho18], as well as the classical case, where it might be necessary to consider half-integer weights for the torus action. This happens already with the square root of the virtual canonical bundle in the classical case.

Remark 9.5. Observe that $\mathcal{N}$ does not have connected stabilizers, so we have to perform the derived stabilizer reduction procedure with non-connected stabilizers in this case.

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