Unconditional Uniqueness for the Cubic Gross-Pitaevskii Hierarchy via Quantum de Finetti

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Abstract

We present a new, simpler proof of the unconditional uniqueness of solutions to the cubic Gross-Pitaevskii hierarchy in $\mathbb{R}^3$. One of the main tools in our analysis is the quantum de Finetti theorem. Our uniqueness result is equivalent to the one established in the celebrated works of Erdős, Schlein, and Yau. © 2015 Wiley Periodicals, Inc.

1 Introduction

In this paper, we give a new proof of unconditional uniqueness of solutions to the cubic Gross-Pitaevskii (GP) hierarchy in $\mathbb{R}^3$. The cubic GP hierarchy is a system of infinitely many coupled linear PDEs describing a Bose gas of infinitely many particles, interacting via repulsive two-body delta interactions in the defocusing case and via attractive two-body delta interactions in the focusing case. In the defocusing case, it emerges from the $N \to \infty$ limit of the BBGKY hierarchy of marginal density matrices for a bosonic N-particle Schrödinger system where the pair interaction potentials tend to a delta distribution as $N \to \infty$. Factorized solutions to GP hierarchies are determined by solutions of the corresponding cubic nonlinear Schrödinger (NLS) equation. In this sense, the NLS is interpreted as the mean field description of an infinite system of interacting bosons in the Gross-Pitaevskii limit.

The derivation of the nonlinear Hartree (NLH) equation from an interacting Bose gas was first given by Hepp in [28]. The BBGKY hierarchy was prominently used in the works of Lanford for the study of classical mechanical systems in the infinite particle limit [33, 34]. Subsequently, the first derivation of the NLH

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via the BBGKY hierarchy was given by Spohn in [43]. More recently this topic was revisited by Fröhlich, Tsai, and Yau in [24], and in the last few years Erdős, Schlein, and Yau have further developed the BBGKY hierarchy approach to the derivation of the NLH and NLS in their landmark works [16–19], which initiated much of the current widespread interest in this research topic.

The proof strategy can be briefly summarized as follows. We consider N bosons in $\mathbb{R}^3$ described by the wave function $\Phi_N \in L^2_{\mathrm{sym}}(\mathbb{R}^{3N})$, where $\Phi_N(x_1, \dots, x_N)$ is symmetric under permutation of particle variables and satisfies the Schrödinger equation

$$i\partial_t \Phi_N = H_N \Phi_N \tag{1.1}$$

with N-body Hamiltonian

$$H_N = \sum_{j=1}^{N} (-\Delta_{x_j}) + \frac{1}{N} \sum_{1 \le i < j \le N} V_N(x_i - x_j). \tag{1.2}$$

The pair interaction potential has the form $V_N(x) = N^{3\beta} V(N^\beta x)$ with $\beta \in (0, 1]$. V is a sufficiently regular, rotationally symmetric pair interaction potential (we will not specify V in detail since the derivation of the NLS is not the topic of this paper). We assume that $\int V(x)dx = 1$ if $\beta < 1$, and that the scattering length corresponding to V has value 1 if $\beta = 1$. Clearly, $V_N \rightharpoonup (\int V\, dx)\delta$ as $N \to \infty$.

For $k = 1, \dots, N-1$, the k-particle marginal density matrices are obtained from

$$\gamma_N^{(k)} := \mathrm{Tr}_{k+1,k+2,\dots,N} |\Phi_N\rangle\langle\Phi_N|, \tag{1.3}$$

where $\mathrm{Tr}_{k+1,k+2,\dots,N}$ denotes the partial trace with respect to the particle variables indexed by $k+1, k+2, \dots, N$. It follows immediately that the property of *admissibility* holds,

$$\gamma_N^{(k)} = \mathrm{Tr}_{k+1}\big(\gamma_N^{(k+1)}\big), \quad k = 1, \dots, N-1 \tag{1.4}$$

for $1 \le k \le N-1$ and that $\mathrm{Tr}(\gamma_N^{(k)}) = \|\Phi_N\|_{L^2}^2 = 1$ for all $k < N$.

The k-particle marginals satisfy the N-particle BBGKY hierarchy

$$\begin{aligned} i\partial_t \gamma_N^{(k)}&(t, \underline{x}_k; \underline{x}'_k) \\ &= -\sum_{j=1}^{k} (\Delta_{x_j} - \Delta_{x'_j}) \gamma_N^{(k)}(t, \underline{x}_k, \underline{x}'_k) \\ &\quad + \frac{1}{N} \sum_{1 \le i < j \le k} [V_N(x_i - x_j) - V_N(x'_i - x'_j)] \gamma_N^{(k)}(t, \underline{x}_k; \underline{x}'_k) + \end{aligned} \tag{1.5}$$

$$+\frac{N-k}{N}\sum_{i=1}^{k}\int dx_{k+1}[V_N(x_i-x_{k+1})-V_N(x_i'-x_{k+1})]$$
$$\cdot\,\gamma_N^{(k+1)}(t,\underline{x}_k,x_{k+1};\underline{x}_k',x_{k+1}).$$

In the limit as $N\to\infty$, solutions $\gamma_N^{(k)}$ to the BBGKY hierarchy tend to solutions $\gamma^{(k)}$ of the cubic, defocusing GP hierarchy, which we introduce below. In [16–19], this is obtained from a weak-$*$ limit in the trace class for $\beta\in(0,1]$. The case $\beta=1$ covered in [16–19] is a major achievement that is much harder than $\beta<1$. Strong convergence in a class of Hilbert-Schmidt type is obtained in [10] for $\beta<\frac{1}{4}$, and subsequently in [13, 14] for $\beta<\frac{2}{3}$.

The cubic GP hierarchy on $\mathbb{R}^3$ for an infinite sequence of bosonic marginal density matrices $(\gamma^{(k)})_{k\in\mathbb{N}}$ is given by the infinite system of coupled linear PDEs

$$i\partial_t\gamma^{(k)}=\sum_{j=1}^{k}\big[-\Delta_{x_j},\gamma^{(k)}\big]+\lambda B_{k+1}\gamma^{(k+1)},\quad k\in\mathbb{N},\tag{1.6}$$

for suitable initial data $(\gamma^{(k)}(0))_{k\in\mathbb{N}}$, where $\gamma^{(k)}(t;\underline{x}_k;\underline{x}_k')$ is fully symmetric under permutations separately of the components of $\underline{x}_k:=(x_1,\dots,x_k)$, and of the components of $\underline{x}_k':=(x_1',\dots,x_k')$. The GP hierarchy is *defocusing* if $\lambda=1$ and *focusing* if $\lambda=-1$ (we are assuming the normalization condition $|\lambda|=1$ for simplicity). The interaction term for the k-particle marginal is defined by

$$B_{k+1}\gamma^{(k+1)}=B_{k+1}^{+}\gamma^{(k+1)}-B_{k+1}^{-}\gamma^{(k+1)},\tag{1.7}$$

where

$$B_{k+1}^{+}\gamma^{(k+1)}=\sum_{j=1}^{k}B_{j;k+1}^{+}\gamma^{(k+1)}\tag{1.8}$$

and

$$B_{k+1}^{-}\gamma^{(k+1)}=\sum_{j=1}^{k}B_{j;k+1}^{-}\gamma^{(k+1)},\tag{1.9}$$

with

$$\begin{aligned}&\big(B_{j;k+1}^{+}\gamma^{(k+1)}\big)(t,x_1,\dots,x_k;x_1',\dots,x_k')\\&\quad=\int dx_{k+1}\,dx_{k+1}'\\&\qquad\cdot\,\delta(x_j-x_{k+1})\delta(x_j-x_{k+1}')\gamma^{(k+1)}(t,x_1,\dots,x_{k+1};x_1',\dots,x_{k+1}')\\&\quad=\gamma^{(k+1)}(t,x_1,\dots,x_j,\dots,x_k,x_j;x_1',\dots,x_k',x_j)\end{aligned}\tag{1.10}$$

and

$$
\begin{aligned}
&\big(B^-_{j;k+1}\gamma^{(k+1)}\big)(t,x_1,\dots,x_k;x'_1,\dots,x'_k)\\
&\quad=\int dx_{k+1}\,dx'_{k+1}\\
&\qquad\cdot\delta(x'_j-x_{k+1})\delta(x'_j-x'_{k+1})\gamma^{(k+1)}(t,x_1,\dots,x_{k+1};x'_1,\dots,x'_{k+1})\\
&\quad=\gamma^{(k+1)}(t,x_1,\dots,x_k,x'_j;x'_1,\dots,x'_j,\dots,x'_k,x'_j).
\end{aligned}
\tag{1.11}
$$

We say that $B^+_{j;k+1}$ *contracts* the triple of variables x_j, x_{k+1}, x'_{k+1} and that $B^-_{j;k+1}$ contracts the triple of variables x'_j, x_{k+1}, x'_{k+1}.

For $\alpha \geq 0$, we define the spaces

$$
\begin{aligned}
\mathfrak{H}^\alpha := \big\{&(\gamma^{(k)})_{k\in\mathbb{N}} \;\big|\\
&\mathrm{Tr}(|S^{(k,\alpha)}[\gamma^{(k)}]|) < M^{2k} \text{ for some constant } M < \infty\big\}
\end{aligned}
\tag{1.12}
$$

where

$$
S^{(k,\alpha)}[\gamma^{(k)}](\underline{x}_k;\underline{x}'_k) := \prod_{j=1}^k (1-\Delta_{x_j})^{\alpha/2}(1-\Delta_{x'_j})^{\alpha/2}\gamma^{(k)}(\underline{x}_k;\underline{x}'_k).
\tag{1.13}
$$

A *mild solution* in the space $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ to the GP hierarchy with initial data $(\gamma^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1$ is a solution of the integral equation

$$
\begin{aligned}
\gamma^{(k)}(t) &= U^{(k)}(t)\gamma^{(k)}(0)\\
&\quad + i\lambda\int_0^t U^{(k)}(t-s)B_{k+1}\gamma^{(k+1)}(s)ds, \quad k\in\mathbb{N},
\end{aligned}
\tag{1.14}
$$

satisfying

$$
\sup_{t\in[0,T)} \mathrm{Tr}(|S^{(k,1)}[\gamma^{(k)}(t)]|) < M^{2k}
\tag{1.15}
$$

for a finite constant M independent of k. Here,

$$
U^{(k)}(t) := \prod_{\ell=1}^k e^{it(\Delta_{x_\ell}-\Delta_{x'_\ell})}
\tag{1.16}
$$

denotes the free k-particle propagator. We note that

$$
e^{it(\Delta_{x_\ell}-\Delta_{x'_\ell})}B^\pm_{j;k+1} = B^\pm_{j;k+1}e^{it(\Delta_{x_\ell}-\Delta_{x'_\ell})}, \quad \ell \notin \{j,k+1\}.
\tag{1.17}
$$

That is, any free propagator $e^{it(\Delta_{x_\ell}-\Delta_{x'_\ell})}$ commutes with $B^\pm_{j;k+1}$ if the variables x_ℓ, x'_ℓ are not affected by $B^\pm_{j;k+1}$.

We remark that given factorized initial data,

$$
\gamma_0^{(k)}(\underline{x}_k;\underline{x}'_k) = \prod_{j=1}^k \phi_0(x_j)\overline{\phi_0(x'_j)},
\tag{1.18}
$$

the condition that $(\gamma^{(k)}(0)) \in \mathfrak{H}^1$ is equivalent to

$$\mathrm{Tr}(|S^{(k,1)}[\gamma^{(k)}(0)]|) = \|\phi_0\|_{H^1}^{2k} < M^{2k}, \quad k \in \mathbb{N}, \tag{1.19}$$

that is, $\|\phi_0\|_{H^1} < M$ for some $M < \infty$. Then, a solution to the GP hierarchy in $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ having these initial data is given by the sequence of *factorized* density matrices

$$\gamma^{(k)}(t; \underline{x}_k; \underline{x}_k') = \prod_{j=1}^{k} \phi_t(x_j)\,\overline{\phi_t(x_j')} \tag{1.20}$$

if the corresponding 1-particle wave function satisfies the cubic NLS

$$i\partial_t\phi_t = -\Delta\phi_t + \lambda|\phi_t|^2\phi_t, \quad \phi_0 \in H^1. \tag{1.21}$$

In this sense, the NLS is interpreted as the mean field description of an infinite system of interacting bosons. The Cauchy problem (1.21) is globally well-posed in the defocusing case $\lambda = 1$ and locally well-posed in the focusing case $\lambda = -1$ [46]. Solutions to (1.21) conserve the L^2-mass $\|\phi_t\|_{L^2_x} = \|\phi_0\|_{L^2_x}$ and the energy,

$$E[\phi_t] = \frac{1}{2}\|\nabla_x\phi_t\|_{L^2_x}^2 + \frac{\lambda}{4}\|\phi_t\|_{L^4_x}^4 = E[\phi_0]. \tag{1.22}$$

In particular, $(\gamma^{(k)})_{k\in\mathbb{N}} \in L^\infty_{t\in[0,T)}\mathfrak{H}^1$ is equivalent to $\|\phi_t\|_{L^\infty_{t\in[0,T)}H^1} < M'$ for some finite constant M'.

The *uniqueness of solutions* to the GP hierarchy in $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ was established by Erdős, Schlein, and Yau in [16–19]. This is a crucial and very involved part of their program to derive the cubic defocusing NLS as the mean field description of a bosonic N-body Schrödinger evolution as $N \to \infty$. Their uniqueness proof uses a sophisticated and extensive construction involving Feynman graph expansions and high-dimensional singular integral estimates. A key ingredient in their proof is a powerful combinatorial method that resolves the problem of the factorial growth of the number of terms in iterated Duhamel expansions; we outline it in Section 4.1.

Subsequently, Klainerman and Machedon [32] gave a much shorter proof of the uniqueness of solutions to the GP hierarchy satisfying

$$\Big\|\Big[\prod_{j=1}^{k}|\nabla_{x_j}||\nabla_{x_j'}|\Big]\gamma^{(k)}\Big\|_{\mathrm{HS}} < c^k, \quad k \in \mathbb{N}, \tag{1.23}$$

for the Hilbert-Schmidt norms

$$\begin{aligned}\|\gamma^{(k)}\|_{\mathrm{HS}} &:= (\mathrm{Tr}(|\gamma^{(k)}|^2))^{1/2}\\ &= \bigg(\int_{\mathbb{R}^{3k}\times\mathbb{R}^{3k}} \big|\gamma^{(k)}(\underline{x}_k; \underline{x}_k')\big|^2 d\underline{x}_k\, d\underline{x}_k'\bigg)^{1/2},\end{aligned} \tag{1.24}$$

but conditional on the a priori assumption that

$$\Big\| \Big\| \Big[\prod_{j=1}^{k} |\nabla_{x_j}| \, |\nabla_{x_j'}| \Big] B^{\pm}_{k+1} \gamma^{(k+1)} \Big\|_{\mathrm{HS}} \Big\|_{L^1_{t\in[0,T)}} < C^k \tag{1.25}$$

holds for some finite constants c, C independent of k. We will refer to (1.25) as the *Klainerman-Machedon condition*. Their approach uses techniques from the analysis of dispersive nonlinear PDEs, together with the combinatorial method of Erdős, Schlein, and Yau [16–19], which Klainerman and Machedon presented as the "boardgame argument." Starting with the work [31] for the cubic GP hierarchy on $\mathbb{R}^2$ and $\mathbb{T}^2$, the approach of Klainerman and Machedon was used by various authors for the derivation of the NLS from interacting Bose gases [9, 10, 12–14, 31, 47]. The method of Klainerman and Machedon also inspired the analysis of the Cauchy problem for the GP hierarchy that was initiated in [8] and then continued (e.g., in [12, 25]).

The derivation of nonlinear dispersive PDEs, such as the nonlinear Schrödinger (NLS) or nonlinear Hartree (NLH) equations, from many-body quantum dynamics is a very active research topic and has been approached by many authors in a variety of ways; see [16–20, 31, 42] and the references therein, and also [1, 3–5, 21–23, 26–28, 40, 41].

This problem is closely related to the phenomenon of Bose-Einstein condensation (BEC) in systems of interacting bosons, which was first experimentally verified in 1995 [6, 15]. For the mathematical study of BEC, we refer to [2, 36–39] and the references therein.

2 Statement of Main Results

The only currently available proof of unconditional uniqueness of solutions in $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ to the cubic GP hierarchy in $\mathbb{R}^3$ is given in the celebrated works of Erdős, Schlein, and Yau [16–19], using an involved construction based on Feynman graph expansions and high-dimensional singular integral estimates. The purpose of this paper is to present a new, simpler proof. We note that this paper contains several extensive example calculations and detailed explanations of background material for the benefit of the reader, but the actual core of our proof, given in Sections 7 and 8, is short. We expect that our methods can be extended to solutions in $L^\infty_{t\in[0,T)}\mathfrak{H}^s$ for some values $s < 1$ and to GP hierarchies in $\mathbb{R}^n$ with n other than 3.

2.1 Prerequisites

A key tool in our proof is the *quantum de Finetti* theorem, which is a quantum analogue of the Hewitt-Savage theorem in probability theory [29]. The strong version is due to Hudson-Moody and Stormer [30, 45] and applies to sequences of density matrices that are *admissible*, i.e.,

$$\gamma^{(k)} = \mathrm{Tr}_{k+1}(\gamma^{(k+1)}) \quad \forall k \in \mathbb{N}, \tag{2.1}$$

similarly to (1.4). We quote it in the formulation presented by Lewin, Nam, and Rougerie in [35] ([30, 45] state it in the C^*-algebraic context).

THEOREM 2.1 (Strong Quantum de Finetti Theorem [30, 35, 45]). *Let $\mathcal{H}$ be any separable Hilbert space and let $\mathcal{H}^k = \bigotimes_{\text{sym}}^k \mathcal{H}$ denote the corresponding bosonic k-particle space. Let Γ denote a collection of admissible bosonic density matrices on $\mathcal{H}$, i.e.,*

$$\Gamma = (\gamma^{(1)}, \gamma^{(2)}, \dots)$$

with $\gamma^{(k)}$ a nonnegative trace class operator on $\mathcal{H}^k$ and $\gamma^{(k)} = \mathrm{Tr}_{k+1}\, \gamma^{(k+1)}$, where Tr_{k+1} denotes the partial trace over the $(k+1)^{th}$ factor. Then there exists a unique Borel probability measure μ, supported on the unit sphere $\mathbb{S} \subset \mathcal{H}$ and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus 1, such that

$$\gamma^{(k)} = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad \forall k \in \mathbb{N}. \tag{2.2}$$

The limiting hierarchies of marginal density matrices obtained, for each value of the time variable t, via weak-$*$ limits from the BBGKY hierarchy of bosonic N-body Schrödinger systems as in [16–19] do not necessarily satisfy admissibility. A weak version of the quantum de Finetti theorem then still applies; in the form quoted in Theorem 2.2 below, it was recently proven by Lewin, Nam, and Rougerie [35]. Previously Ammari and Nier proved an equivalent result in [3, 4] in the context of N-body boson systems as $N \to \infty$ with less singular interactions than those considered for the GP hierarchy.

THEOREM 2.2 (Weak Quantum de Finetti Theorem [3, 4, 35]). *Let $\mathcal{H}$ be any separable Hilbert space, and let $\mathcal{H}^k = \bigotimes_{\text{sym}}^k \mathcal{H}$ denote the corresponding bosonic k-particle space. Assume that $\gamma_N^{(N)}$ is an arbitrary sequence of mixed states on $\mathcal{H}^N$, $N \in \mathbb{N}$, satisfying $\gamma_N^{(N)} \geq 0$ and $\mathrm{Tr}_{\mathcal{H}^N}(\gamma_N^{(N)}) = 1$, and assume that its k-particle marginals have weak-$*$ limits*

$$\gamma_N^{(k)} := \mathrm{Tr}_{k+1,\dots,N}\big(\gamma_N^{(N)}\big) \rightharpoonup^* \gamma^{(k)}, \quad N \to \infty, \tag{2.3}$$

in the trace class on $\mathcal{H}^k$ for all $k \geq 1$ (here, $\mathrm{Tr}_{k+1,\dots,N}(\gamma_N^{(N)})$ denotes the partial trace in , the $(k+1)^{st}$ up to N^{th} component). Then, there exists a unique Borel probability measure μ on the unit ball $\mathcal{B} \subset \mathcal{H}$ and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus 1 such that (2.2) holds for all $k \geq 0$.

In our case, we consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$. For a detailed discussion of the strong and weak quantum de Finetti theorem, we refer to [35], where the notions of strong and weak quantum de Finetti were introduced.

2.2 Main Results

Our main result is a new proof of the *unconditional* uniqueness of solutions to the GP hierarchy in $L^\infty_{t\in[0,T)}\mathfrak{H}^1$. Note that the property $(\gamma^{(k)})_{k\in\mathbb{N}} \in L^\infty_{t\in[0,T)}\mathfrak{H}^1$ implies that $\gamma^{(k)}(t) = \int d\mu_t(\phi)(|\phi\rangle\langle\phi|)^{\otimes k}$, where the measure μ_t has bounded support in $H^1(\mathbb{R}^3)$; this is explained in Lemma 4.5 below.

THEOREM 2.3. *Let $(\gamma^{(k)}(t))_{k\in\mathbb{N}}$ be a mild solution in $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ to the (de)focusing cubic GP hierarchy in $\mathbb{R}^3$ with initial data $(\gamma^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1$, which is either admissible or obtained at each t from a weak-$*$ limit as described in Theorem 2.2. Then, $(\gamma^{(k)})_{k\in\mathbb{N}}$ is the unique solution for the given initial data.*

Moreover, assume that the initial data $(\gamma^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1$ satisfy

$$\gamma^{(k)}(0) = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad \forall k \in \mathbb{N} \tag{2.4}$$

(as guaranteed by Theorems 2.1 and 2.2) where μ is a Borel probability measure supported either on the unit sphere or on the unit ball in $L^2(\mathbb{R}^3)$, and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus 1. Then,

$$\gamma^{(k)}(t) = \int d\mu(\phi)(|S_t(\phi)\rangle\langle S_t(\phi)|)^{\otimes k} \quad \forall k \in \mathbb{N}, \tag{2.5}$$

where $S_t : \phi \mapsto \phi_t$ is the flow map of the cubic (de)focusing NLS for $t \in [0,T)$. That is, ϕ_t satisfies (1.21) with initial data ϕ.

Our proof of uniqueness uses the Erdős-Schlein-Yau combinatorial method [16–19], in boardgame form as presented by Klainerman-Machedon in [32]. However, we do *not* use the Klainerman-Machedon condition (1.25), but instead apply the quantum de Finetti theorems. The uniqueness established in Theorem 2.3 is unconditional and equivalent to the one proven in the celebrated works of Erdős, Schlein, and Yau [16–19].

Uniqueness of Strong Solutions

Our next result addresses the uniqueness of strong solutions and shows the strength of de Finetti for the GP hierarchy. We consider strong solutions to the GP hierarchy $\Gamma \in C^1([0,T),\mathfrak{H}^1)$ with $\partial_t\Gamma \in C([0,T),\mathfrak{H}^{-1})$ in $\mathbb{R}^d$ where $d = 1,2,3$ (see [7] for a discussion on the level of the NLS). The quantum de Finetti theorem can be used for a short direct proof of uniqueness in this case, along the lines of Spohn's argument for the Vlasov hierarchy [44].

THEOREM 2.4. *Let $d \in \{1,2,3\}$ and $\Gamma = (\gamma^{(k)})$. Let*

$$\widehat{\Delta}\Gamma := \Big(\sum_{j=1}^{k}[\Delta_{x_j},\gamma^{(k)}]\Big)_{k\in\mathbb{N}} \quad \textit{and} \quad \widehat{B}\Gamma := (B_{k+1}\gamma^{(k+1)}).$$

Let $\Gamma(t) = (\gamma^{(k)}(t))$ be a strong solution of the GP hierarchy in $C^1([0,T),\mathfrak{H}^1)$ on $\mathbb{R}^d$, which is either admissible or obtained at each t from a weak-$*$ limit as *described in Theorem 2.2. Then,*

$$i\partial_t \Gamma(t) = \widehat{L}\Gamma(t) \in C([0,T),\mathfrak{H}^{-1}), \quad \widehat{L} = -\widehat{\Delta} + \lambda \widehat{B}, \quad \lambda \in \{1,-1\}, \tag{2.6}$$

and $\Gamma(t)$ *is the unique solution for the given initial data.*

3 Proof of Theorem 2.4

Let $\Gamma_\phi := ((|\phi\rangle\langle\phi|)^{\otimes k})_{k\in\mathbb{N}}$ for brevity. We know that $\int d\nu(\phi)\Gamma_{S_t(\phi)}$ solves (2.6), with $\Gamma_0 = \int d\nu(\phi)\Gamma_\phi$, if $S_t(\phi)$ is the flow map of the cubic (de)focusing NLS

$$i\partial_t \phi_t = -\Delta\phi_t + \lambda|\phi_t|^2\phi_t \tag{3.1}$$

with initial data $\phi_0 \in H^1(\mathbb{R}^d)$.

Since by assumption $\Gamma(t)$ is either admissible or obtained at each t from a weak-$*$ limit as described in Theorem 2.2, the de Finetti theorem implies that for every $t \geq 0$ there exists a Borel probability measure μ_t on $L^2(\mathbb{R}^d)$ such that

$$\Gamma(t) = \int d\mu_t(\phi)\Gamma_\phi. \tag{3.2}$$

Furthermore, we note that for any $s \geq 0$ and an arbitrary small $\eta > 0$,

$$\mathrm{Tr}\big(\big|S^{(k+1,s-\frac{d}{2}-\eta)}[(\widehat{B}\Gamma)^{(k)}]\big|\big) \leq C\,\mathrm{Tr}\big(\big|S^{(k+1,s)}[\Gamma^{(k+1)}]\big|\big). \tag{3.3}$$

This inequality was proven in [11] (it corresponds to a generalized Sobolev inequality for density matrices). Moreover, it is evident that for any $s \geq 0$,

$$\mathrm{Tr}\big(\big|S^{(k,s-2)}[(\widehat{\Delta}\Gamma)^{(k)}]\big|\big) \leq \mathrm{Tr}\big(\big|S^{(k,s)}[\Gamma^{(k)}]\big|\big), \quad k \in \mathbb{N}. \tag{3.4}$$

Since $\frac{d}{2} + \eta \leq 2$ for $d \leq 3$ in (3.3), we find that by setting $s = 1$ in (3.3) and (3.4),

$$\mathrm{Tr}\big(\big|S^{(k,-1)}[(\widehat{L}\Gamma)^{(k)}]\big|\big) < C\,\mathrm{Tr}\big(\big|S^{(k,1)}[\Gamma^{(k)}]\big|\big) < C^k, \quad k \in \mathbb{N}, \tag{3.5}$$

or in other words, $\widehat{L}\Gamma \in C([0,T),\mathfrak{H}^{-1})$. Here, the second inequality follows from $\Gamma \in C^1([0,T),\mathfrak{H}^1)$; see (1.19). Since $\Gamma(t)$ is a strong solution of (2.6) in $C^1([0,T),\mathfrak{H}^1)$, we obtain that

$$i\partial_t \Gamma^{(k)}(t) = \lim_{h\to 0}\frac{1}{h}(\Gamma^{(k)}(t+h) - \Gamma^{(k)}(t)) = (\widehat{L}\Gamma)^{(k)}(t), \tag{3.6}$$

hence $\partial_t\Gamma \in C([0,T),\mathfrak{H}^{-1})$. Then, indeed,

$$\begin{aligned} \frac{d}{dt}\int d\mu_t(\phi)\Gamma_\phi &= \lim_{h\to 0}\frac{1}{h}\Big(\int d\mu_{t+h}(\phi)\Gamma_\phi - \int d\mu_t(\phi)\Gamma_\phi\Big) \\ &= -i\int \widehat{L}\Gamma_\phi\, d\mu_t(\phi) \end{aligned} \tag{3.7}$$

holds in $C([0,T),\mathfrak{H}^{-1})$. In this sense, μ_t is differentiable, with derivative given by the operator $\widehat{L}$.

In analogy to Spohn's argument in [44], we can show that the measure μ_t induces a flow on the unit ball that satisfies the GP-equation (3.1). To this end, we define

$$\widetilde{\Gamma}(t) := \int d\mu_t(\phi)\Gamma_{S_{-t}(\phi)}. \tag{3.8}$$

Differentiating this with respect to t, gives

$$\begin{aligned}
\frac{d}{dt}\widetilde{\Gamma}(t) &= \lim_{h\to 0}\frac{1}{h}\left(\int d\mu_{t+h}(\phi)\Gamma_{S_{-t-h}(\phi)} - \int d\mu_t(\phi)\Gamma_{S_{-t}(\phi)}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\int d\mu_{t+h}(\phi)\Gamma_{S_{-t}(\phi)} - \int d\mu_t(\phi)\Gamma_{S_{-t}(\phi)}\right) &(3.9)\\
&\quad + \lim_{h\to 0}\frac{1}{h}\int d\mu_{t+h}(\phi)(\Gamma_{S_{-t-h}(\phi)} - \Gamma_{S_{-t}(\phi)}) &(3.10)\\
&= -i\int d\mu_t(\phi)\widehat{L}\Gamma_{S_{-t}(\phi)} + i\int d\mu_t(\phi)\widehat{L}\Gamma_{S_{-t}(\phi)} = 0, &(3.11)
\end{aligned}$$

where we applied (3.7) to (3.9) to get the first term in (3.11), and where we applied (3.6) to (3.10) to get the second term in (3.11).

Since the map $\phi \mapsto S_t(\phi)$ is a bijection on the intersection of the unit ball of $L^2(\mathbb{R}^d)$ with $H^1(\mathbb{R}^d)$, we obtain by a simple variable transformation

$$\int d\mu_t(\phi)\Gamma_{S_{-t}(\phi)} = \int d\mu_t(S_t(\phi))\Gamma_\phi. \tag{3.12}$$

Because of (3.11), we find that

$$\widetilde{\Gamma}(t) = \int d\mu_t(S_t(\phi))\Gamma_\phi \tag{3.13}$$

does not depend on t, and by the uniqueness parts of Theorems 2.1 and 2.2, we infer that

$$d\mu_t(S_t(\phi)) = d\mu_0(\phi).$$

Hence, by variable transformation,

$$\Gamma(t) = \int d\mu_t(\phi)\Gamma_\phi = \int d\mu_t(S_t(\phi))\Gamma_{S_t(\phi)} = \int d\mu_0(\phi)\Gamma_{S_t(\phi)}.$$

By the uniqueness parts of Theorems 2.1 and 2.2, $d\nu = d\mu_0$. □

4 Proof of Theorem 2.3

The remainder of this paper is dedicated to the proof of Theorem 2.3. As a preparation, we present three auxiliary combinatorial tools used for the organization of Duhamel expansion terms, in Sections 4.1, 5, and 6. For the convenience of the reader, we will give a detailed survey of background material in these parts and present several detailed example calculations. The core of our proof is contained in Sections 7 and 8 and is short.

4.1 The Erdős-Schlein-Yau Combinatorial Method in Boardgame Form

To prove uniqueness, we will show that the trace norm of $\gamma^{(k)}$ (instead of the Hilbert-Schmidt norm of $S^{(k,1)}[\gamma^{(k)}]$ as in [32]) is 0 if the initial data is 0 (see Section 4.2 for details). We will use the powerful combinatorial method of Erdős, Schlein, and Yau [16–19], which was presented in an elegant and accessible form by Klainerman and Machedon in [32] as a "boardgame argument."

To begin with, we consider the r-fold iterate of the Duhamel formula (1.14) for $\gamma^{(k)}$, with initial data $\gamma_0^{(k)} = 0$, for some arbitrary $r \in \mathbb{N}$:

$$\gamma^{(k)}(t) = (i\lambda)^r \int_{t \ge t_1 \ge \cdots \ge t_r} dt_1 \cdots dt_r U^{(k)}(t-t_1) B_{k+1} U^{(k+1)}(t_1 - t_2) \cdots U^{(k+r-1)}(t_{r-1} - t_r) B_{k+r} \gamma^{(k+r)}(t_r)$$

$$(4.1) \qquad =: \int_{t \ge t_1 \ge \cdots \ge t_r} dt_1 \cdots dt_r J^k(\underline{t}_r), \quad \underline{t}_r := (t_1, \dots, t_r).$$

We will prove the following main lemma.

LEMMA 4.1. *Assume that $(\gamma^{(k)}(t))$ is a mild solution to the cubic GP hierarchy* (1.6) *with initial data $\gamma^{(k)}(0) = 0$ for all k, which is either admissible or obtained at each t from a weak-$*$ limit as in Theorem* 2.2. *Moreover, assume that*

$$(4.2) \qquad \sup_{t \in [0,T)} \mathrm{Tr}(|S^{(k,1)}[\gamma^{(k)}(t)]|) < M^{2k}, \quad k \in \mathbb{N},$$

holds for some finite constant M independent of k and t.

Then, for $t \in [0, T)$, the estimate

$$(4.3) \qquad \mathrm{Tr}(|\gamma^{(k)}(t)|) < 2M^{2k-2}(2CM^4T)^{(r+1)/2}$$

holds. In particular, the right-hand side converges to 0 *in the limit as $r \to \infty$ for $T < (2CM^4)^{-1}$ (independently of k) for every $k \in \mathbb{N}$.*

This main lemma implies that

$$(4.4) \qquad \mathrm{Tr}(|\gamma^{(k)}(t)|) = 0, \quad t \in [0, T),$$

and thus that $\gamma^{(k)}(t) = 0$ for $t \in [0, T)$. Hence, uniqueness holds.

A key difficulty in this approach stems from the fact that the interaction operator $B_{\ell+1}$ is the sum of $O(\ell)$ terms; therefore (4.1) contains $O(\frac{(k+r-1)!}{(k-1)!}) = O(r!)$ terms. The boardgame argument allows one to control this rapid increase of the number of terms as $r \to \infty$ by using the fact that the ordered time integrals $t_1 \ge t_2 \ge \cdots \ge t_r$ extend over a simplex of volume $O(\frac{1}{r!})$. We give a short summary of the method.

Recalling that $B_{\ell+1} = \sum_{j=1}^{\ell} B_{j;\ell+1}$, we write

$$J^k(\underline{t}_r) = \sum_{\rho\in\mathcal{M}_{k,r}} J^k(\rho;\underline{t}_r), \tag{4.5}$$

where

$$\begin{aligned} J^k(\rho;\underline{t}_r) := (i\lambda)^r U^{(k)}(t-t_1) B_{\rho(k+1),k+1} U^{(k+1)}(t_1-t_2) \\ \cdots U^{(k+\ell-1)}(t_{\ell-1}-t_\ell) B_{\rho(k+\ell),k+\ell} \\ \cdots U^{(k+r-1)}(t_{r-1}-t_r) B_{\rho(k+r),k+r} \gamma^{(k+r)}(t_r), \end{aligned} \tag{4.6}$$

and ρ is a map

$$\begin{gathered} \rho : \{k+1, r+2, \dots, k+r\} \to \{1, 2, \dots, k+r-1\}, \\ \rho(2) = 1, \quad \rho(j) < j \quad \forall j. \end{gathered} \tag{4.7}$$

Here $\mathcal{M}_{k,r}$ denotes the set of all such mappings ρ.

We observe that each map ρ can be represented by highlighting one nonzero entry $B_{\rho(k+\ell),k+\ell}$ in each column of an $(k+r-1)\times r$ matrix $[A_{i,\ell}]$ with entries $A_{i,\ell} = B_{i,k+\ell}$ for $i < k+\ell$, and $A_{i,\ell} = 0$ for $i \geq k+\ell$. As an example, consider

$$\begin{bmatrix} \mathbf{B_{1,k+1}} & B_{1,k+2} & \cdots & \cdots & \cdots & \mathbf{B_{1,k+r}} \\ \vdots & \mathbf{B_{2,k+2}} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \cdots & \mathbf{B_{\rho(k+\ell),k+\ell}} & \cdots & \cdots \\ B_{k,k+1} & B_{k,k+2} & \cdots & \cdots & \cdots & \cdots \\ 0 & B_{k+1,k+2} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \cdots & \cdots & \cdots \\ \vdots & & \ddots & \ddots & \cdots & \cdots \\ \vdots & & & \ddots & \ddots & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & B_{k+r-1,k+r} \end{bmatrix}. \tag{4.8}$$

Then,

$$\gamma^{(k)}(t) = \sum_{\rho\in\mathcal{M}_{k,r}} \int_{t\geq t_1\geq\cdots\geq t_r} J^k(\rho,\underline{t}_r)\,dt_1\cdots dt_r, \tag{4.9}$$

where the time domains are given by the same simplex $\{t > t_1 > \cdots > t_r\} \subset [0,t]^r$ for all integrals in the sum over ρ.

We next consider the integrals with permuted time integration orders

$$I(\rho,\pi) = \int_{t \geq t_{\pi(1)} \geq \cdots \geq t_{\pi(r)}} J^k(\rho; \underline{t}_r)\, dt_1 \cdots dt_r, \tag{4.10}$$

where π is a permutation of $\{1, 2, \ldots, r\}$. This corresponds to replacing the simplex $\{t > t_1 > \cdots > t_r\} \subset [0,t]^r$ by an isometric image in $[0,t]^r$. One can associate to $I(\rho,\pi)$ the matrix

$$\begin{bmatrix} t_{\pi^{-1}(1)} & t_{\pi^{-1}(2)} & \cdots & \cdots & t_{\pi^{-1}(r)} \\ \mathbf{B_{1,k+1}} & B_{1,k+2} & \cdots & \cdots & \mathbf{B_{1,k+r}} \\ \vdots & \mathbf{B_{2,k+2}} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ B_{k,k+1} & B_{k,k+2} & \cdots & \cdots & \vdots \\ 0 & B_{k+1,k+2} & \cdots & \cdots & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & B_{k+r-1,k+r} \end{bmatrix}$$

whose columns are labeled 1 through r and whose rows are labeled $0, 1, \ldots, k + r - 1$, and where the highlighted entries correspond to $B_{\rho(k+\ell),k+\ell}$.

Using the combinatorial method in [16–19] in the form presented in [32], a *boardgame* is introduced on the set of such matrices. An *acceptable move* is characterized as follows: If $\rho(k+\ell) < \rho(k+\ell-1)$, the player is allowed to do the following three changes at the same time:

- exchange the highlights in columns ℓ and $\ell + 1$,
- exchange the highlights in rows $k + \ell - 1$ and $k + \ell$,
- exchange $t_{\pi^{-1}(\ell)}$ and $t_{\pi^{-1}(\ell+1)}$.

We note that the rows $k + \ell$ and $k + \ell + 1$ do not necessarily contain highlights. A main property of the integrals $I(\rho,\pi)$ is *invariance under acceptable moves* [16–19, 32]:

LEMMA 4.2. *If (ρ,π) is transformed into (ρ',π') by an acceptable move, then $I(\rho,\pi) = I(\rho',\pi')$.*

We say that a matrix of the type (4.8) is in *upper echelon form if each highlighted entry in a row is to the left of each highlighted entry in a lower row*. For example, the following matrix is in upper echelon form (with $k = 1$ and $r = 4$):

$$\begin{bmatrix} \mathbf{B_{1,2}} & B_{1,3} & B_{1,4} & B_{1,5} \\ 0 & \mathbf{B_{2,3}} & B_{2,4} & B_{2,5} \\ 0 & 0 & \mathbf{B_{3,4}} & \mathbf{B_{3,5}} \\ 0 & 0 & 0 & B_{4,5} \end{bmatrix}.$$

Then, the following *normal form* property holds [16–19, 32]:

LEMMA 4.3. *For each matrix in $\mathcal{M}_{k,r}$, there is a finite number of acceptable moves that transforms the matrix into upper echelon form. Moreover, let $C_{k,r}$ denote the number of upper echelon matrices of size $(k+r-1)\times r$. Then,*

$$C_{k,r} \le 2^{k+r}. \tag{4.11}$$

Let $\mathcal{N}_{k,r}$ denote the subset of matrices in $\mathcal{M}_{k,r}$ that are in upper echelon form. Let σ account for a matrix in $\mathcal{N}_{k,r}$. We write $\rho \sim \sigma$ if the matrix corresponding to ρ can be transformed into that corresponding to σ in finitely many acceptable moves. We note that σ satisfies the same properties (4.7) as ρ, but in addition,

$$\sigma(j) \le \sigma(j') \quad \forall j < j'. \tag{4.12}$$

Then, the following key theorem holds [16–19, 32]:

THEOREM 4.4. *Suppose $\sigma \in \mathcal{N}_{k,r}$. Then there exists a subset of $[0,t]^r$, denoted by $D(\sigma,t)$, such that*

$$\sum_{\rho\sim\sigma} \int_{t\ge t_1\ge\cdots\ge t_r} J^k(\rho;\underline{t}_r)\,dt_1\cdots dt_r = \int_{D(\sigma,t)} J^k(\sigma;\underline{t}_r)\,dt_1\cdots dt_r. \tag{4.13}$$

We remark that $D(\sigma,t)$ is the union of all simplices $\{t > t_{\pi(1)} > \cdots > t_{\pi(r)}\} \subset [0,t]^r$ obtained under acceptable moves for the fixed upper echelon form σ; notably, the interiors of these simplices are all pairwise disjoint. We emphasize that the main point of Theorem 4.4 is the reduction of a sum of $O(r!)$ terms to a sum of $O(C^r)$ terms. This concludes our summary of the Erdős-Schlein-Yau combinatorial method [16–19], formulated in boardgame form following Klainerman-Machedon [32].

4.2 Setup of the Proof

We now give a precise formulation of the framework in which we will prove Theorem 2.3. Let us assume that we have two positive semidefinite solutions $(\gamma_j^{(k)}(t))_{k\in\mathbb{N}} \in L^\infty_{t\in[0,T)}\mathfrak{H}^1$ satisfying the same initial data, $(\gamma_1^{(k)}(0))_{k\in\mathbb{N}} = (\gamma_2^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1$. Then,

$$\gamma^{(k)}(t) := \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t) \quad k \in \mathbb{N}, \tag{4.14}$$

is a solution to the GP hierarchy with initial data $\gamma^{(k)}(0) = 0\ \forall k \in \mathbb{N}$, and it suffices to prove that $\gamma^{(k)}(t) = 0\ \forall k \in \mathbb{N}$, and for all $t \in [0,T)$. This is due to the linearity of the GP hierarchy.

We note that $\gamma^{(k)}$, as a difference of positive semidefinite marginal density matrices, is not in general positive semidefinite.

From the assumptions of Theorem 2.3, we have that

$$\sup_{t\in[0,T)} \mathrm{Tr}\big(\big|S^{(k,1)}\big[\gamma_i^{(k)}(t)\big]\big|\big) < M^{2k}, \quad k \in \mathbb{N}, \ i = 1,2, \tag{4.15}$$

for some finite constant M independent of k and t.

To ensure the applicability of Theorems 2.1 and 2.2, we note that if

$$\gamma_j^{(k)} = \mathrm{Tr}_{k+1}\big(\gamma_j^{(k+1)}\big) \quad \forall k \in \mathbb{N}, \quad j = 1, 2, \tag{4.16}$$

are admissible, it follows immediately that $(\gamma^{(k)})_{k\in\mathbb{N}}$ is admissible. Moreover, if both $(\gamma_1^{(k)})$ and $(\gamma_2^{(k)})$ are obtained from a weak-$*$ limit, then so is $(\gamma^{(k)})$.

Thus, from Theorems 2.1 and 2.2, we have that

$$\begin{aligned} \gamma_j^{(k)}(t) &= \int d\mu_t^{(j)}(\phi)(|\phi\rangle\langle\phi|)^{\otimes k}, \quad j = 1, 2, \\ \gamma^{(k)}(t) &= \int d\widetilde{\mu}_t(\phi)(|\phi\rangle\langle\phi|)^{\otimes k}, \end{aligned} \tag{4.17}$$

where $\widetilde{\mu}_t := \mu_t^{(1)} - \mu_t^{(2)}$ is the difference of two probability measures on the unit ball in $L^2(\mathbb{R}^3)$. We remark that (4.15) is equivalent to

$$\int d\mu_t^{(j)}(\phi)\|\phi\|_{H^1}^{2k} < M^{2k}, \quad j = 1, 2, \tag{4.18}$$

for all $k \in \mathbb{N}$, where $H^1 = \{f \in L^2(\mathbb{R}^3) \mid \|\langle\nabla_x\rangle f\|_{L^2} < \infty\}$ and $\langle\nabla\rangle := \sqrt{1-\Delta}$.

LEMMA 4.5. *Let μ be a Borel probability measure in $L^2(\mathbb{R}^3)$, and assume that*

$$\int d\mu(\phi)\|\phi\|_{H^1}^{2k} \le M^{2k} \tag{4.19}$$

holds for some finite constant $M > 0$ and all $k \in \mathbb{N}$. Then,

$$\mu(\{\phi \in L^2(\mathbb{R}^3) \mid \|\phi\|_{H^1} > M\}) = 0. \tag{4.20}$$

PROOF. From Chebyshev's inequality, we have that

$$\mu(\{\phi \in L^2(\mathbb{R}^3) \mid \|\phi\|_{H^1} > \lambda\}) \le \frac{1}{\lambda^{2k}} \int d\mu(\phi)\|\phi\|_{H^1}^{2k} \le \frac{M^{2k}}{\lambda^{2k}} \tag{4.21}$$

for any $k \ge 0$. Evidently, for $\lambda > M$, the right-hand side tends to 0 when $k \to \infty$. $\square$

From here on, we will consider the representation of the expansion (4.1) for $\gamma^{(k)}(t)$ in upper echelon normal form, given by the right-hand side of (4.13). Then,

$$\begin{aligned} \gamma^{(k)}(t) = \sum_{\sigma\in\mathcal{N}_{k,r}} \int_{D(\sigma,t)} & dt_1 \cdots dt_r U^{(k)}(t-t_1) B_{\sigma(k+1),k+1} U^{(k+1)}(t_1 - t_2) \\ & \cdots U^{(k+r-1)}(t_{r-1} - t_r) B_{\sigma(k+r),k+r} \gamma^{(k+r)}(t_r)\,. \end{aligned} \tag{4.22}$$

The sum with respect to σ extends over all nonequivalent upper echelon forms.

Using the quantum de Finetti theorem (by which we henceforth refer to either the strong or the weak version), we obtain

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{N}_{k,r}} \int_{D(\sigma,t)} dt_1 \cdots dt_r \int d\widetilde{\mu}_{t_r}(\phi)\, J^k(\sigma; t, t_1, \dots, t_r), \tag{4.23}$$

where

$$\begin{aligned} &J^k(\sigma; t, t_1, \dots, t_r; \underline{x}_k; \underline{x}'_k) \\ &\quad = \big(U^{(k)}(t - t_1) B_{\sigma(k+1),k+1} U^{(k+1)}(t_1 - t_2) \\ &\qquad \cdots B_{\sigma(k+\ell),k+\ell} U^{(k+\ell)}(t_\ell - t_{\ell+1}) \\ &\qquad \cdot B_{\sigma(k+\ell+1),k+\ell+1} U^{(k+\ell+1)}(t_{\ell+1} - t_{\ell+2}) \\ &\qquad \cdots U^{(k+r-1)}(t_{r-1} - t_r) B_{\sigma(k+r),k+r} \big(|\phi\rangle\langle\phi|\big)^{\otimes(k+r)}\big)(\underline{x}_k; \underline{x}'_k). \end{aligned} \tag{4.24}$$

Here, we may think of the time variable t_ℓ as being attached to the interaction operator $B_{\sigma(k+\ell),k+\ell}$. For *fixed* ϕ, we note that since

$$(|\phi\rangle\langle\phi|)^{\otimes(k+r)}(\underline{x}_{k+r}; \underline{x}'_{k+r}) = \prod_{i=1}^{k+r} (|\phi\rangle\langle\phi|)(x_i; x'_i) \tag{4.25}$$

is given by a product of 1-particle kernels, it follows that

$$J^k(\sigma; t, t_1, \dots, t_r; \underline{x}_k; \underline{x}'_k) = \prod_{j=1}^{k} J^1_j(\sigma_j; t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}}; x_j; x'_j) \tag{4.26}$$

likewise has product form for each fixed σ. This is because in (4.24) the operators $B_{\sigma(k+\ell),k+\ell}$ and $U^{(k+\ell)}(t_\ell - t_{\ell+1})$ each map products of 1-particle kernels to products of 1-particle kernels (but the operators $B_{\sigma(k+\ell),k+\ell}$ in general do not preserve positive semidefiniteness). Each 1-particle kernel J^1_j can be written as a Duhamel expansion in itself, with interaction operators inherited from those appearing in J^k. We label the interaction operators in J^1_j "internally" with σ_j, $j = 1, \dots, k$ (which are automatically in upper echelon form relative to J^1_j). More details are given in Section 5 below.

For a fixed k, the number of inequivalent echelon forms is bounded by C^r using Lemma 4.3. Hence,

$$\begin{aligned} \operatorname{Tr}(|\gamma^{(k)}|) \le C^r \sum_{i=1,2} \sup_\sigma \int_{[0,t]^r} dt_1 \cdots dt_r \int d\mu^{(i)}_{t_r}(\phi) \\ \cdot \prod_{j=1}^{k} \operatorname{Tr}\big(\big|J^1_j(\sigma_j; t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}})\big|\big). \end{aligned} \tag{4.27}$$

The time variable $t_{\ell_{j,\alpha}}$ corresponds to the one attached to the α^{th} interaction operator (counting from the left) appearing in the factor J^1_j (every $t_{\ell_{j,\alpha}}$ corresponds uniquely to one of the time variables t_ℓ in (4.24)). Based on the expression (4.27), our goal is to prove the estimate

$$\mathrm{Tr}(|\gamma^{(k)}(t)|) < 2\, M^{2k-2}\, (2CM^4T)^{(r+1)/2} \tag{4.28}$$

asserted in Lemma 4.1; it implies that for any $k \in \mathbb{N}$, the right-hand side tends to 0 as $r \to \infty$ for $t \in [0,T)$ and sufficiently small $T > 0$ (independently of k). Since r is arbitrary, this implies that the left-hand side equals 0, thus establishing uniqueness. By iterating this argument on the union of intervals $[0,T) \cup [T,2T) \cup \cdots$, uniqueness extends to the entire time of existence for a given solution. We note that in (4.27), the distinction between focusing and defocusing GP hierarchy has disappeared, since $|\lambda| = 1$ in both cases.

5 Binary Tree Graphs

Because every interaction operator contracts precisely two factors, one can conveniently organize the above expansions for J^k and J^1_j with the help of binary tree graphs for arbitrary values of k and r. For the convenience of the reader, we first discuss the factorization (4.26) with an example.

5.1 An Example for $k = 3$

As an example, let $k = 3$ and $r = 4$, and let us consider

$$J^3(\sigma; t, t_1, \dots, t_4) = U^{(3)}_{0,1} B_{2,4} U^{(4)}_{1,2} B_{2,5} U^{(5)}_{2,3} B_{3,6} U^{(6)}_{3,4} B_{5,7} (|\phi\rangle\langle\phi|)^{\otimes 7} \tag{5.1}$$

where $U^{(j)}_{i,i'} := U^{(j)}(t_i - t_{i'})$ with $t_0 := t$, and where σ corresponds to the upper echelon matrix

$$\begin{bmatrix} B_{1,4} & B_{1,5} & B_{1,6} & B_{1,7} \\ \mathbf{B_{2,4}} & \mathbf{B_{2,5}} & B_{2,6} & B_{2,7} \\ B_{3,4} & B_{3,5} & \mathbf{B_{3,6}} & B_{3,7} \\ 0 & B_{4,5} & B_{4,6} & B_{4,7} \\ 0 & 0 & B_{5,6} & \mathbf{B_{5,7}} \\ 0 & 0 & 0 & B_{6,7} \end{bmatrix}. \tag{5.2}$$

We now read off from this matrix which terms are "connected" via contractions, starting with the rightmost interaction operator. Although all seven factors in $(|\phi\rangle\langle\phi|)^{\otimes 7}$ are indistinguishable, we enumerate the factors and write the product in the form $\otimes_{i=1}^7 u_i$, ordered with increasing index i (where $u_i = |\phi\rangle\langle\phi|$ for every $i = 1, \dots, 7$).

- Clearly, $B_{5,7}$ contracts the factors u_5 and u_7, and acts trivially as the identity on all other factors u_i,

$$B_{5,7}\Big(\bigotimes_{i=1}^{7} u_i\Big) = \Big(\bigotimes_{i=1}^{4} u_i\Big) \otimes \Theta_4 \otimes u_6, \tag{5.3}$$

where

$$\Theta_4 := B_{1,2}(u_5 \otimes u_7). \tag{5.4}$$

The index α in Θ_α associates it to the α^{th} interaction operator from the left in (5.1) (in the case of Θ_4, the fourth interaction operator is given by $B_{5,7}$).

- The interaction operator $B_{3,6}$ contracts $U_{3,4}^{(1)}u_3$ and $U_{3,4}^{(1)}u_6$, while it leaves all remaining factors untouched. In particular, it does not affect Θ_4.

$$B_{3,6}U_{3,4}^{(6)}\big((5.3)\big) = \big(U_{3,4}^{(2)}(u_1 \otimes u_2)\big) \otimes \Theta_3 \otimes \big(U_{3,4}^{(1)}u_4\big) \otimes (U_{3,4}^{(1)}\Theta_4) \tag{5.5}$$

where

$$\Theta_3 := B_{1,2}\big(U_{3,4}^{(2)}(u_3 \otimes u_6)\big). \tag{5.6}$$

- The interaction operator $B_{2,5}$ contracts $U_{2,4}^{(1)}u_2$ with $U_{2,4}^{(1)}\Theta_4$ (where we used the group property $U_{2,3}^{(j)}U_{3,4}^{(j)} = U_{2,4}^{(j)}$) corresponding to the second and fifth factor in (5.5), while it leaves all remaining factors untouched.

$$B_{2,5}U_{2,3}^{(5)}\big((5.5)\big) = \big(U_{2,4}^{(1)}u_1\big) \otimes \Theta_2 \otimes \big(U_{2,3}^{(1)}\Theta_3\big) \otimes \big(U_{2,4}^{(1)}u_4\big) \tag{5.7}$$

where

$$\Theta_2 := B_{1,2}\big(\big(U_{2,4}^{(1)}u_2\big) \otimes \big(U_{2,4}^{(1)}\Theta_4\big)\big). \tag{5.8}$$

- Finally, the interaction operator $B_{2,4}$ contracts $U_{1,2}^{(1)}\Theta_2$ with $(U_{1,4}^{(1)}u_4)$ corresponding to the second and fourth factor in (5.7), while leaving all other factors untouched,

$$B_{2,4}U_{1,2}^{(4)}\big((5.7)\big) = \big(U_{1,4}^{(1)}u_1\big) \otimes \Theta_1 \otimes \big(U_{1,3}^{(1)}\Theta_3\big) \tag{5.9}$$

where

$$\Theta_1 := B_{1,2}\big(\big(U_{1,2}^{(1)}\Theta_2\big) \otimes \big(U_{1,4}^{(1)}u_4\big)\big). \tag{5.10}$$

In conclusion, we have found the factorized expression for (5.1),

$$J^3 = \underbrace{\big(U_{0,4}^{(1)}u_1\big)}_{=J_1^1} \otimes \underbrace{\big(U_{0,1}^{(1)}\Theta_1\big)}_{=J_2^1} \otimes \underbrace{\big(U_{0,3}^{(1)}\Theta_3\big)}_{=J_3^1}. \tag{5.11}$$

We may now write the factors J_j^1 as one-particle matrices and substitute back $u_i = |\phi\rangle\langle\phi|$ for all $i = 1, \dots, 7$.

- J_1^1 corresponds to a free propagation without any interaction operators,

$$J_1^1 = U_{0,4}^{(1)}|\phi\rangle\langle\phi| \tag{5.12}$$

- Moreover,

$$J_2^1 = U_{0,1}^{(1)}B_{1,2}U_{1,2}^{(2)}B_{1,3}U_{2,4}^{(1)}B_{3,4}(|\phi\rangle\langle\phi|)^{\otimes 4} \tag{5.13}$$

where the interaction operators correspond to $B_{2,4}$, $B_{2,5}$, and $B_{5,7}$; they are reindexed in a manner that leaves the connectivity structure between contractions invariant. The labeling of interaction operators $B_{\sigma_2(\ell),\ell}$ here is obtained from a labeling function σ_2 (corresponding to σ_j in (4.26)) where $\sigma_2(2) = 1$, $\sigma_2(3) = 1$, and $\sigma_2(4) = 3$.

- Finally,

$$J_3^1 = U_{0,3}^{(1)} B_{1,2} U_{3,4}^{(2)} (|\phi\rangle\langle\phi|)^{\otimes 2} \tag{5.14}$$

where the interaction operator corresponds to $B_{3,6}$ and can be labeled with $\sigma_3(2) = 1$.

We observe that for $\ell < \ell'$, the interaction operators $B_{\sigma(\ell),\ell}$ and $B_{\sigma(\ell'),\ell'}$ in J^3 (which are highlighted in (5.2)) belong to the same factor J_j^1 if either $\sigma(\ell) = \sigma(\ell')$ or $\ell = \sigma(\ell')$. In this case, we can think of them as being "connected"; below, we will introduce binary tree graphs that encode this connectivity structure.

This example also illustrates how the internal labeling functions σ_j in (4.26) are deduced from the "global" labeling function σ. In the sense outlined above, we may think of σ_j as the restriction of σ to J_j^1.

We note that in J_1^1 and J_3^1, there is a free propagator applied to each ϕ; this will allow for a straightforward application of Strichartz estimates to control contractions due to interaction operators. However, the term J_2^1 involves the contraction of factors $|\phi\rangle\langle\phi|$ without any free propagation term in between. We will call the factor J_2^1 *distinguished* while J_1^1 and J_3^1 are regular.

5.2 Definition of Binary Trees

We now introduce binary tree graphs as a bookkeeping device to keep track of the complicated contraction structures imposed by the interaction operators inside the iterated Duhamel formula (4.24).

To this end, we associate (4.24) to the union of k disjoint binary tree graphs, $(\tau_j)_{j=1}^k$. We note that these appear as "skeleton graphs" for the more complicated graphs in [16–19]. We assign:

- An *internal vertex* v_ℓ, $\ell = 1, \ldots, r$, to each operator $B_{\sigma(k+\ell),k+\ell}$. Accordingly, the time variable t_ℓ in (4.24) is thought of as being attached to the vertex v_ℓ.
- A *root vertex* w_j, $j = 1, \ldots, k$, to each factor $J_j^1(\cdots; x_j; x_j')$ in (4.26).
- A *leaf vertex* u_i, $i = 1, \ldots, k + r$, to the factor $(|\phi\rangle\langle\phi|)(x_i; x_i')$ in (4.25).

For the sake of concreteness, we draw graphs as follows: We consider the strip in $(x, y) \in \mathbb{R}^2$ given by $x \in [0, 1]$. We draw all root vertices $(w_j)_{j=1}^k$, ordered vertically, on the line $x = 0$, all internal vertices $(v_\ell)_{\ell=1}^r$ in the region $x \in (0, 1)$, where $v_{\ell'}$ is on the right of v_ℓ if $\ell' > \ell$. Finally, we draw all leaf vertices $(u_i)_{i=1}^{k+r}$, ordered vertically, on the line $x = 1$.

Next, we introduce the equivalence relation "$\sim$" of *connectivity* between vertices. Between any pair of connected vertices, we draw a connecting line, which we refer to as an *edge*:

- Let v_ℓ be the internal vertex with the smallest value of ℓ such that $\sigma(\ell) = j$; then we say that v_ℓ is connected to the root vertex w_j, that is, $w_j \sim v_\ell$.
- If there is no internal vertex connected to w_j, we draw an edge connecting w_j to the leaf vertex u_j and say that they are connected, $w_j \sim u_j$.
- Given $k < \ell \leq k + r$, if there exists $\ell' > \ell$ such that $\ell = \sigma(\ell')$ or $\sigma(\ell) = \sigma(\ell')$, we say that $v_\ell \sim v_{\ell'}$ are *connected.*

  v_ℓ is then called a *parent vertex* of $v_{\ell'}$, and $v_{\ell'}$ is called a *child vertex* of v_ℓ. We denote the two child vertices of v_ℓ by $v_{\kappa_-(\ell)}$ and $v_{\kappa_+(\ell)}$, using the condition $\kappa_-(\ell) < \kappa_+(\ell)$.

  If there exists no internal vertex $v_{\ell'}$ with $\ell' > \ell$ such that $\ell = \sigma(\ell')$, we say that v_ℓ is connected to the leaf vertex u_ℓ, $v_\ell \sim u_\ell$; if there exists no internal vertex $v_{\ell'}$ with $\ell' > \ell$ such that $\sigma(\ell) = \sigma(\ell')$, we say that v_ℓ is connected to the leaf vertex $u_\sigma(\ell)$, $v_\ell \sim u_{\sigma(\ell)}$. In these cases, v_ℓ is the *parent vertex* of u_ℓ (or $u_{\sigma(\ell)}$), and u_ℓ (or $u_{\sigma(\ell)}$) is a *child vertex* of v_ℓ.

This implies that every internal vertex has precisely two child vertices, which can be either internal or leaf vertices (they do not need to be of the same type). Every root vertex has precisely one child vertex, which could be of internal or leaf type. Every internal or leaf vertex has exactly one parent vertex.

We conclude that the graph thus obtained is the disjoint union of k binary trees, which we denote by $(\tau_j)_{j=1}^k$, where the root of τ_j is the root vertex w_j (if $w_j \sim u_j$ without internal vertices in between, then the binary tree consists trivially only of a single edge connecting one root and one leaf vertex).

We say that the tree τ_j is *distinguished* if $v_r \in \tau_j$ and *regular* if $v_r \notin \tau_j$. We call the two leaf vertices connected to v_r *distinguished leaf vertices* and all others *regular leaf vertices*. Clearly, there are $k - 1$ regular trees, and one distinguished tree in this construction. In Figure 5.1, the tree graph corresponding to the example in Section 5 is shown.

6 The Distinguished Tree Graph

In this section, we further refine the combinatorial organization of terms corresponding to the distinguished tree. We note that regular trees can be treated in a similar manner, with obvious modifications. Let τ_j denote the distinguished tree graph. We assume that it contains m_j internal vertices $(v_{\ell_{j,\alpha}})_{\alpha=1}^{m_j}$ and $m_j + 1$ leaf vertices $(u_{j,i})_{i=1}^{m_j+1}$. We recall that the internal vertices are enumerated with $\alpha \in \{1, \ldots, m_j\}$, where $\alpha = m_j$ is the distinguished vertex, and that α corresponds to the interaction operator $B_{\sigma_j(\alpha+1),\alpha+1}$ (notice the shift by 1 in the index) in (6.3). For notational simplicity, we will from here on label leaf vertices with $\alpha \in \{m_j + 1, \ldots, 2m_j + 2\}$ (corresponding to $u_{j,\alpha-m_j}$) and will often refer to the vertex $v_{j,\alpha}$ by its label α.

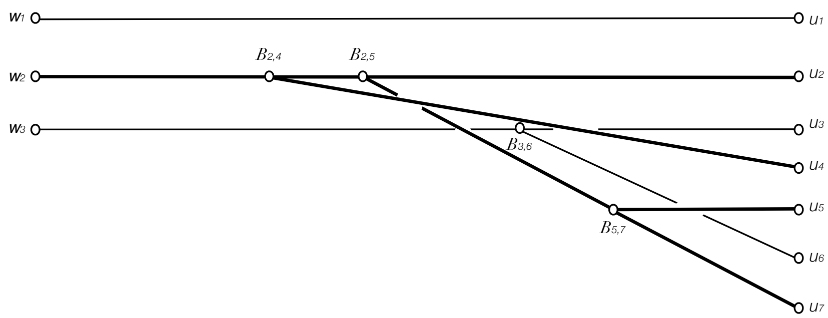

FIGURE 5.1. The disjoint union of three tree graphs τ_j, $j = 1, 2, 3$, corresponding to the example discussed in Section 5.1 for $k = 3$, $r = 4$, and

$$J^3(\sigma; t, t_1, \ldots, t_4) = U_{0,1}^{(3)} B_{2,4} U_{1,2}^{(4)} B_{2,5} U_{2,3}^{(5)} B_{3,6} U_{3,4}^{(6)} B_{5,7} (|\phi\rangle\langle\phi|)^{\otimes 7}.$$

The root vertex w_j belongs to the tree τ_j, $j = 1, 2, 3$. The internal vertices correspond to $v_1 \sim B_{2,4}$, $v_2 \sim B_{2,5}$, $v_3 \sim B_{3,6}$, and $v_4 \sim B_{5,7}$. The leaf vertices u_5 and u_7 and the internal vertex $v_4 \sim B_{5,7}$ are distinguished. The distinguished tree τ_2 is drawn with thick edges.

To determine the contribution to (4.26) corresponding to τ_j, we use the commutativity relation (1.17) and straightforwardly find that

$$\begin{aligned} &J_j^1(\sigma_j; t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,m_j}}; x_j; x_j') \\ &\quad = U^{(1)}(t - t_{\ell_{j,1}}) \cdots U^{(1)}(t_{\ell_{j,1}-1} - t_{\ell_{j,1}}) B_{\sigma_j(2),2} \\ &\qquad \cdots B_{\sigma_j(\alpha),\alpha} U^{(\alpha)}(t_{\ell_{j,\alpha-1}} - t_{\ell_{j,\alpha-1}+1}) \\ &\qquad \cdots U^{(\alpha)}(t_{\ell_{j,\alpha}-1} - t_{\ell_{j,\alpha}}) B_{\sigma_j(\alpha+1),\alpha+1} \\ &\qquad \cdots U^{(m_j)}(t_{\ell_{j,m_j}-1} - t_{\ell_{j,m_j}}) B_{\sigma_j(m_j+1),m_j+1} \big(|\phi\rangle\langle\phi|\big)^{\otimes(m_j+1)}. \end{aligned} \tag{6.1}$$

Here, the interaction operators $B_{\sigma_j(\alpha),\alpha}$ are adapted to the 1-particle kernel J_j^1, and $(\sigma_j)_{j=1}^{m_j}$ is an "internal" labeling of the interaction operators that preserves the structure of τ_j. In this sense, σ_j corresponds to the restriction of σ to the tree τ_j. Clearly, $\sigma_j(2) = 1$.

Between any two consecutive interaction operators $B_{\sigma_j(\alpha),\alpha}$ and $B_{\sigma_j(\alpha+1),\alpha+1}$, with $\alpha < m_j$, there is a composition of $\ell_{j,\alpha} - \ell_{j,\alpha-1}$ free propagators at consecutive time steps, so that

$$U^{(\alpha)}(t_{\ell_{j,\alpha-1}} - t_{\ell_{j,\alpha-1}+1}) \cdots U^{(\alpha)}(t_{\ell_{j,\alpha}-1} - t_{\ell_{j,\alpha}}) = U^{(\alpha)}(t_{\ell_{j,\alpha-1}} - t_{\ell_{j,\alpha}}), \tag{6.2}$$

due to the group property of the free propagators. Hence, (6.1) reduces to

$$
\begin{aligned}
&J_j^1(\sigma_j; t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}}) \\
&\quad = U^{(1)}(t - t_{\ell_{j,1}}) B_{1,2} \\
&\qquad \cdots B_{\sigma_j(\alpha),\alpha} U^{(\alpha)}(t_{\ell_{j,\alpha-1}} - t_{\ell_{j,\alpha}}) B_{\sigma_j(\alpha+1),\alpha+1} \\
&\qquad \cdots U^{(m_j)}(t_{\ell_{j,m_j-1}} - t_{\ell_{j,m_j}}) B_{\sigma_j(m_j+1),m_j+1} (|\phi\rangle\langle\phi|)^{\otimes(m_j+1)}
\end{aligned}
\tag{6.3}
$$

where $\ell_{j,m_j} = r$. We observe that on the last line, there is no free propagator in front of $\big(|\phi\rangle\langle\phi|\big)^{\otimes(m_j+1)}$ (since τ_j is distinguished). As a consequence, the Strichartz estimate cannot be applied in the last step. Resolving this issue is the main task of the construction presented in subsequent sections.

Our goal is to bound

$$
\int_{[0,T)^{m_j-1}} dt_{\ell_{j,1}} \cdots dt_{\ell_{j,m_j-1}} \operatorname{Tr}\Big(\big|J_j^1(\sigma_j; t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}})\big|\Big). \tag{6.4}
$$

We will employ a recursion that takes into account the structure of interactions and free evolutions occurring between interactions. We will explain the strategy based on an example in the next section.

6.1 Example Calculation for a Distinguished Tree

We consider an example of a distinguished tree, which we obtain from setting $k = 1$, $r = 3$ in (4.23) (if $k = 1$, there is only one tree, and it is necessarily distinguished). From (4.23) and (4.24), we have

$$
\begin{aligned}
\gamma^{(1)}(t) = (i\lambda)^3 \sum_{\sigma \in \mathcal{N}_{1,3}} \int_{D(\sigma,t)} & dt_1\, dt_2\, dt_3 \\
& \int d\widetilde{\mu}_{t_3}(\phi) U^{(1)}(t - t_1) B_{1,2} U^{(2)}(t_1 - t_2) \\
& \cdot B_{\sigma(3),3} U^{(3)}(t_2 - t_3) B_{\sigma(4),4} \big(|\phi\rangle\langle\phi|\big)^{\otimes 4},
\end{aligned}
\tag{6.5}
$$

where $D(\sigma, t) \subseteq [0, t]^3$. For a fixed σ (with, say, $\sigma(3) = 2$ and $\sigma(4) = 3$), we consider, as an example, the contribution to the bound (4.27) of the form

$$
\begin{aligned}
\int_{[0,T)^3} dt_1\, dt_2\, dt_3 \int d\mu_{t_3}^{(i)}(\phi) \operatorname{Tr}\Big(\big|\big(U^{(1)}(t - t_1) B_{1,2} U^{(2)}(t_1 - t_2) \\
B_{2,3} U^{(3)}(t_2 - t_3) B_{3,4} (|\phi\rangle\langle\phi|)^{\otimes 4}\big|\Big),
\end{aligned}
\tag{6.6}
$$

where $t \in [0, T)$, and noting that $|i\lambda| = 1$. In Figure 6.1, another example of a distinguished tree is shown.

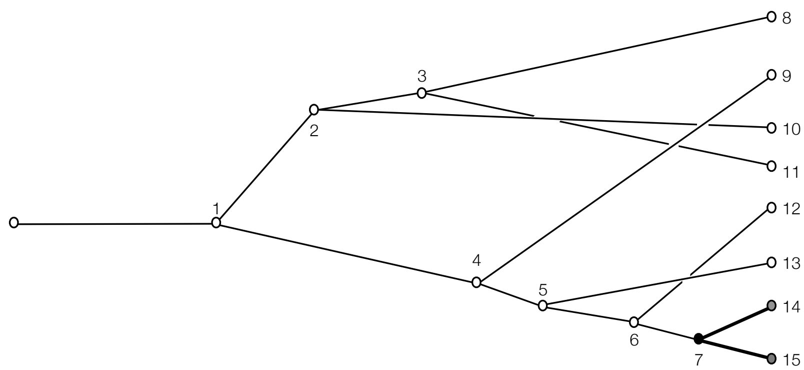

FIGURE 6.1. An example of a distinguished tree τ_j with $m_j = 7$. The number next to an internal vertex corresponds to its label $\alpha \in \{1, \dots, 7\}$, ordered increasingly from left to right. The vertex with label α corresponds to $B_{\sigma_j(\alpha+1),\alpha+1}$ (note the shift by 1 in the index). The time variable attached to it is $t_{\ell_{j,\alpha}}$. The leftmost vertex is the root vertex, while the rightmost vertices are leaf vertices labeled with $\alpha \in \{8, 15\}$, corresponding to $u_{j,\alpha-7}$. The leaf vertices 8 to 13 are regular, while the leaf vertices 14, 15 are distinguished. The distinguished internal vertex has label 7. Here, for example, $\kappa_-(1) = 2$, $\kappa_+(1) = 4$, $\kappa_+(\kappa_+(\kappa_+(\kappa_+(1)))) = 7$, $\kappa_-(2) = 3$. This tree corresponds to the term

$$U_{0,1}^{(1)} B_{1,2} U_{1,2}^{(2)} B_{1,3} U_{2,3}^{(3)} B_{1,4} U_{3,4}^{(4)} B_{2,5} U_{4,5}^{(5)} B_{5,6} U_{5,6}^{(6)} B_{5,7} U_{6,7}^{(7)} B_{7,8} (|\phi\rangle\langle\phi|)^{\otimes 8}$$

(in upper echelon form) where $U_{i,j}^{(j)} = U^{(j)}(t_i - t_j)$ and $t_0 = t$.

Recursive Determination of Contraction Structure

Clearly, $(|\phi\rangle\langle\phi|)^{\otimes 4}$ is a product of 1-particle density matrices. We observe that the interaction operators $B_{i,j}$ preserve the product structure (while changing the explicit expressions for each factor) and contract two factors at a time (the i^{th} and the j^{th}). On all other factors, $B_{i,j}$ acts as the identity. Similarly, as in the example of Section 5.1, we introduce kernels Θ_α, $\alpha = 1, \dots, 3$, which account for the contractions performed by $B_{\sigma(\alpha+1),\alpha+1}$, which we write in the normal form

$$\Theta_\alpha(x, x') = \sum_{\beta_\alpha} c^\alpha_{\beta_\alpha} \chi^\alpha_{\beta_\alpha}(x) \overline{\psi^\alpha_{\beta_\alpha}}(x') \tag{6.7}$$

where $\chi^\alpha_{\beta_\alpha}$, $\psi^\alpha_{\beta_\alpha}$ are certain functions that will be recursively determined, and $c^\alpha_{\beta_\alpha}$ are coefficients with values in $\{1, -1\}$.

THE KERNEL Θ_3. We start at the last interaction operator $B_{3,4}$ in (6.6). It acts nontrivially only on the third and fourth factor in $(|\phi\rangle\langle\phi|)^{\otimes 4}$,

$$B_{3,4}(|\phi\rangle\langle\phi|)^{\otimes 4} = (|\phi\rangle\langle\phi|)^{\otimes 2} \otimes \Theta_3. \tag{6.8}$$

The kernel Θ_3 is obtained from contracting a 2-particle density matrix to a 1-particle density matrix via the interaction operator $B_{1,2}$ (which acts on a 2-particle kernel $f(x,y;x',y')$ by $(B_{1,2}f)(x,x') = f(x,x;x',x) - f(x,x';x',x')$),

$$
\begin{aligned}
\Theta_3(x,x') &:= B_{1,2}\big((|\phi\rangle\langle\phi|)^{\otimes 2}\big)(x,x') = \widetilde{\psi}(x)\overline{\phi(x')} - \phi(x)\overline{\widetilde{\psi}(x')} \\
&=: \sum_{\beta_3=1}^{2} c^3_{\beta_3}\chi^3_{\beta_3}(x)\overline{\psi^3_{\beta_3}}(x')
\end{aligned}
\tag{6.9}
$$

where

$$
\widetilde{\psi} := |\phi|^2\phi. \tag{6.10}
$$

Here, we have $c^3_1 = 1$, $c^3_2 = -1$, $\chi^3_1 = \widetilde{\psi}$, $\chi^3_2 = \phi$, $\psi^3_1 = \phi$, $\psi^3_2 = \widetilde{\psi}$.

The *main difficulty in estimating* (6.6) stems from the fact that the term $\widetilde{\psi} = |\phi|^2\phi$ can only be controlled in L^2, where by Sobolev embedding, $\|\widetilde{\psi}\|_{L^2} \le C\|\phi\|^3_{H^1}$, which can then be controlled by (4.18); see (6.24) below. Our objective thus is to apply the triangle inequality to the trace norm inside (6.6) and to recursively "propagate" the resulting L^2-norm through all intermediate terms until we reach $\widetilde{\psi}$; see (6.15) below. We remark that if $\|\widetilde{\psi}\|_{H^1}$ could be controlled by $\|\phi\|_{H^1}$ (which is not the case), a straightforward application of the method of [32] would suffice to carry out our analysis.

We now reinterpret $\widetilde{\psi}$ in (6.9) as a function that is *independent* of ϕ, $\overline{\phi}$. Only at the end of our analysis, we will substitute $\widetilde{\psi} := |\phi|^2\phi$. We call a factor $\chi^\alpha_{\beta_\alpha}$, $\psi^\alpha_{\beta_\alpha}$ in the sum (6.7) *distinguished* if it is a function of $\widetilde{\psi}$. In the first step, it is clear that for every β_3, only one out of the two factors $\chi^3_{\beta_3}$, $\psi^3_{\beta_3}$ in the sum (6.9) is distinguished (and in fact equal to $\widetilde{\psi}$). The property of being distinguished then propagates from there; i.e., in the next step the distinguished term is the one containing the distinguished term as a factor from the previous step, and so on.

THE KERNEL Θ_2. Next, we consider the terms contracted by $B_{2,3}$ in (6.6),

$$
B_{2,3}U^{(3)}(t_2-t_3)\big((|\phi\rangle\langle\phi|)^{\otimes 2}\otimes\Theta_3\big) = (U^{(1)}(t_2-t_3)|\phi\rangle\langle\phi|)\otimes\Theta_2, \tag{6.11}
$$

using (6.8), which defines the kernel

$$
\begin{aligned}
&\Theta_2(x,x') \\
&\quad = B_{1,2}\big((U^{(1)}(t_2-t_3)|\phi\rangle\langle\phi|)\otimes(U^{(1)}(t_2-t_3)\Theta_3)\big)(x,x') \\
&\quad = (U_{2,3}\phi)(x)\overline{(U_{2,3}\phi)(x')}\sum_{\beta_3=1}^{2} c^3_{\beta_3}\big[(U_{2,3}\chi^3_{\beta_3})(x)\overline{(U_{2,3}\psi^3_{\beta_3})}(x) \\
&\qquad\qquad\qquad\qquad - (U_{2,3}\chi^3_{\beta_3})(x')\overline{(U_{2,3}\psi^3_{\beta_3})}(x')\big] \\
&\quad =: \sum_{\beta_2=1}^{4} c^2_{\beta_2}\chi^2_{\beta_2}(x)\overline{\psi^2_{\beta_2}}(x'),
\end{aligned}
\tag{6.12}
$$

where

$$U_{i,j} := e^{i(t_i - t_j)\Delta}. \tag{6.13}$$

Since for every β_3, only one out of the two factors $\chi^3_{\beta_3}$, $\psi^3_{\beta_3}$ is distinguished, it follows from (6.12) that for every $\beta_2 \in \{1, \dots, 4\}$, only one out of the two factors $\chi^2_{\beta_2}$, $\psi^2_{\beta_2}$ is distinguished. The coefficients $c^2_{\beta_2}$ again have values in $\{1, -1\}$.

THE KERNEL Θ_1. Finally, we consider the terms contracted by $B_{1,2}$ in (6.6), corresponding to

$$\begin{aligned}
&\Theta_1(x, x') \\
&\quad = B_{1,2}\big((U^{(1)}(t_1 - t_2)U^{(1)}(t_2 - t_3)|\phi\rangle\langle\phi|) \\
&\qquad \otimes (U^{(1)}(t_1 - t_2)\Theta_2)\big)(x, x') \\
&\quad = (U_{1,3}\phi)(x)\overline{(U_{1,3}\phi)(x')} \sum_{\beta_2=1}^{4} c^2_{\beta_2}\big[(U_{1,2}\chi^2_{\beta_2})(x)\overline{(U_{1,2}\psi^2_{\beta_2})}(x) \\
&\qquad\qquad - (U_{1,2}\chi^2_{\beta_2})(x')\overline{(U_{1,2}\psi^2_{\beta_2})}(x')\big] \\
&\quad =: \sum_{\beta_1=1}^{8} c^1_{\beta_1}\chi^1_{\beta_1}(x)\overline{\psi^1_{\beta_1}}(x').
\end{aligned} \tag{6.14}$$

Again, since for every β_2, only one out of the two functions $\chi^2_{\beta_2}$, $\psi^2_{\beta_2}$ is distinguished, it follows that for every $\beta_1 \in \{1, \dots, 8\}$, only one out of the two functions $\chi^1_{\beta_1}$, $\psi^1_{\beta_1}$ is distinguished. The coefficients $c^1_{\beta_1}$ again have values in $\{1, -1\}$.

Recursive Bounds

We may now return to (6.6) and perform the following recursive bounds with respect to time integration:

INTEGRAL IN t_1. Applying Cauchy-Schwarz with respect to the integral in t_1 and the triangle inequality for the trace norm, we obtain that

$$\begin{aligned}
(6.6) &= \int_{[0,T)^3} dt_1\, dt_2\, dt_3 \int d\mu^{(i)}_{t_3}(\phi)\, \mathrm{Tr}(|U^{(1)}(t - t_1)\Theta_1|) \\
&\le \sum_{\beta_1=1}^{8} T^{1/2} \int_{[0,T)^2} dt_2\, dt_3 \int d\mu^{(i)}_{t_3}(\phi) \big\| \|\chi^1_{\beta_1}\|_{L^2_x} \|\psi^1_{\beta_1}\|_{L^2_x} \big\|_{L^2_{t_1 \in [0,T)}},
\end{aligned} \tag{6.15}$$

using that $|c^1_{\beta_1}| = 1$. From (6.14), we see that given $\beta_1 \in \{1, \dots, 8\}$, there exists β_2 such that

$$\begin{aligned}
\chi^1_{\beta_1}(x) &= (U_{1,3}\phi)(x), \\
\psi^1_{\beta_1}(x) &= (U_{1,3}\phi)(x)\overline{(U_{1,2}\chi^2_{\beta_2})}(x)(U_{1,2}\psi^2_{\beta_2})(x)
\end{aligned} \tag{6.16}$$

(or with a cubic expressions for $\chi^1_{\beta_1}$ and a linear expression for $\psi^1_{\beta_1}$). Therefore,

$$\big\| \|\chi^1_{\beta_1}\|_{L^2_x} \|\psi^1_{\beta_1}\|_{L^2_x} \big\|_{L^2_{t_1\in[0,T)}} = \|\phi\|_{L^2_x} \big\| (U_{1,3}\phi)(x)\overline{(U_{1,2}\chi^2_{\beta_2})}(x)(U_{1,2}\psi^2_{\beta_2})(x)\big\|_{L^2_{t_1\in[0,T)}L^2_x}, \tag{6.17}$$

using that $U_{1,3}$ is unitary and that ϕ does not depend on t_1.

Next, we observe that

$$\begin{aligned} &\big\| (e^{it\Delta} f_1)(x)\overline{(e^{it\Delta} f_2)(x)}(e^{it\Delta} f_3)(x)\big\|_{L^2_t(\mathbb{R})L^2_x(\mathbb{R}^3)} \\ &\quad\le \|e^{it\Delta} f_1\|_{L^\infty_t L^6_x} \|e^{it\Delta} f_2\|_{L^\infty_t L^6_x} \|e^{it\Delta} f_3\|_{L^2_t L^6_x} \\ &\quad\le C\|f_1\|_{H^1_x}\|f_2\|_{H^1_x}\|f_3\|_{L^2_x} \end{aligned} \tag{6.18}$$

using the Hölder inequality, the Sobolev inequality, and the Strichartz estimate $\|e^{it\Delta} f\|_{L^2_t L^6_x} \le C\|f\|_{L^2}$ for the free Schrödinger evolution. We make the important observation that in (6.18), we can place the L^2_x-norm on any of the three functions f_j, $j = 1, 2, 3$, and not only on f_3. Similarly, if a derivative is included,

$$\begin{aligned} &\big\| \nabla_x((e^{it\Delta} f_1)(x)\overline{(e^{it\Delta} f_2)}(x)(e^{it\Delta} f_3)(x))\big\|_{L^2_t(\mathbb{R})L^2_x(\mathbb{R}^3)} \\ &\quad\le \sum_{j=1}^{3} \|e^{it\Delta}\nabla_x f_j\|_{L^2_t L^6_x} \prod_{\substack{1\le i\le 3\\ i\ne j}} \|e^{it\Delta} f_i\|_{L^\infty_t L^6_x} \\ &\quad\le C\, \|f_1\|_{H^1_x}\|f_2\|_{H^1_x}\|f_3\|_{H^1_x}, \end{aligned} \tag{6.19}$$

which, together with (6.18), implies that

$$\big\| (e^{it\Delta} f_1)(x)\overline{(e^{it\Delta} f_2)}(x)(e^{it\Delta} f_3)(x))\big\|_{L^2_t(\mathbb{R})H^1_x(\mathbb{R}^3)} \le C\prod_{j=1}^{3}\|f_j\|_{H^1_x}. \tag{6.20}$$

Only one of the factors $\chi^2_{\beta_2}$, $\psi^2_{\beta_2}$ is distinguished, say for instance $\psi^2_{\beta_2}$. We then use (6.18) in such a way that the L^2_x-norm is applied to this term. All terms in (6.15) can be treated in the same manner, thus obtaining

$$(6.6) \le CT^{1/2}\sum_{\beta_1=1}^{8}\int_{[0,T)^2} dt_2\, dt_3 \int d\mu^{(i)}_{t_3}(\phi)\|\phi\|^2_{H^1_x}\|\chi^2_{\beta_2}\|_{H^1_x}\|\psi^2_{\beta_2}\|_{L^2_x}, \tag{6.21}$$

where the indices β_2 depend on β_1.

Next, we use the defining relation (6.12) for the functions $\chi^2_{\beta_2}$, $\psi^2_{\beta_2}$, and consider the integral in t_2.

INTEGRAL IN t_2. By assumption, the factor $\psi^2_{\beta_2}$ is distinguished, while $\chi^2_{\beta_2}$ is not. Moreover, one of the functions $\chi^2_{\beta_2}$, $\psi^2_{\beta_2}$ is linear, while the other one is a cubic expression in the functions after the second equality sign in (6.12) (the distinguished factor could be either). Our goal is to bound the distinguished factor in L^2. From comparing terms in (6.12), one possible combination is

$$\begin{aligned}\chi^2_{\beta_2}(x) &= (U_{2,3}\phi)(x),\\ \psi^2_{\beta_2}(x) &= (U_{2,3}\phi)(x)\overline{(U_{2,3}\chi^3_{\beta_3})}(x)(U_{2,3}\psi^3_{\beta_3})(x),\end{aligned} \tag{6.22}$$

that is, the distinguished factor $\psi^2_{\beta_2}$ is a cubic expression. We apply Cauchy-Schwarz in the t_2-integral in such a way that the $L^2_{t_2}$-norm falls on the cubic term. (If, on the other hand, $\chi^2_{\beta_2}$ is the cubic term, we use Cauchy-Schwarz in t_2 to get $\|\psi^2_{\beta_2}\|_{L^\infty_{t_2\in[0,T)}L^2_x}$ and $\|\chi^2_{\beta_2}\|_{L^2_{t_2\in[0,T)}H^1_x} \le C\|\phi\|^3_{H^1_x}$ from (6.20).) We then get

$$\begin{aligned}(6.21) &\le CT\sum_{\beta_1=1}^{8}\int_{[0,T)} dt_3\int d\mu^{(i)}_{t_3}(\phi)\|\phi\|^2_{H^1_x}\|\chi^2_{\beta_2}\|_{L^\infty_{t_2\in[0,T)}H^1_x}\\ &\qquad\qquad\cdot\|\psi^2_{\beta_2}\|_{L^2_{t_2\in[0,T)}L^2_x}\\ &= CT\sum_{\beta_1=1}^{8}\int_{[0,T)} dt_3\int d\mu^{(i)}_{t_3}(\phi)\|\phi\|^3_{H^1_x}\\ &\qquad\cdot\big\|(U_{2,3}\phi)(x)\overline{(U_{2,3}\chi^3_{\beta_3})}(x)\big(U_{2,3}\psi^3_{\beta_3}\big)(x)\big\|_{L^2_{t_2\in[0,T)}L^2_x},\end{aligned} \tag{6.23}$$

where only one of the three factors inside the norm on the last line is distinguished. We may assume it is $\psi^3_{\beta_3}$. By comparing terms in (6.9), we then find that $\psi^3_{\beta_3} = \widetilde{\psi}$, and $\chi^3_{\beta_3} = \phi$. We then apply (6.18) again, and use the L^2_x-bound for $\psi^3_{\beta_3} = \widetilde{\psi}$. At this point, we substitute $\widetilde{\psi} = |\phi|^2\phi$.

USING DE FINETTI FOR THE LAST STEP. Subsequently, we obtain

$$\begin{aligned}(6.6) &\le CT\sum_{\beta_1=1}^{8}\int_{[0,T)} dt_3\int d\mu^{(i)}_{t_3}(\phi)\|\phi\|^5_{H^1}\|\widetilde{\psi}\|_{L^2_x}\\ &\le 8CT^2\sup_{t_3\in[0,T)}\int d\mu^{(i)}_{t_3}(\phi)\|\phi\|^8_{H^1}\\ &\le 8CT^2M^4,\end{aligned} \tag{6.24}$$

where we used $\|\widetilde{\psi}\|_{L^2_x} \le C\|\phi\|^3_{H^1}$ from Sobolev embedding, and the bound (4.18) related to the de Finetti theorem, which is uniform in t_3. This is the desired estimate in our example calculation.

The strategy presented in this example can be applied in the general case.

6.2 Recursive Definition of Kernels at Vertices

As in the above example, we recursively assign a kernel Θ_α to each vertex α. As the root of this induction, we associate the kernel

$$\Theta_\alpha(x;x') := \phi(x)\overline{\phi(x')} \tag{6.25}$$

to the leaf vertex with label $\alpha \in \{m_j+1,\dots,2m_j+2\}$ (corresponding to $u_{j,\alpha-m_j}$).

In the first recursion step, we determine Θ_{m_j} at the distinguished vertex $\alpha = m_j$ from the term on the last line of (6.3), given by

$$\begin{aligned} B_{\sigma_j(m_j+1),m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(m_j+1)} = (|\phi\rangle\langle\phi|)^{\otimes(\sigma(m_j+1)-1)} \otimes \Theta_{m_j} \\ \otimes (|\phi\rangle\langle\phi|)^{\otimes(m_j+1-\sigma(m_j+1)-1)} \end{aligned} \tag{6.26}$$

where

$$\Theta_{m_j}(x;x') := \widetilde{\psi}(x)\overline{\phi(x')} - \phi(x)\overline{\widetilde{\psi}(x')} \tag{6.27}$$

with $\widetilde{\psi}$ as in (6.10). It is obtained from contracting the two copies of $|\phi\rangle\langle\phi|$ at the two leaf vertices $\kappa_-(m_j)$, $\kappa_+(m_j)$ which have m_j as their parent vertex.

For the induction step, we let $\alpha \in \{1,\dots,m_j-1\}$ label a regular internal vertex and assume that the kernels $\Theta_{\alpha'}$ have been determined for all $\alpha' > \alpha$. Let $\kappa_-(\alpha)$, $\kappa_+(\alpha)$ label the two child vertices (of internal or leaf type) of α,

$$\sigma_j(\alpha) = \sigma_j(\kappa_-(\alpha)), \quad \alpha = \sigma_j(\kappa_+(\alpha)). \tag{6.28}$$

Then, by the induction assumption, $\Theta_{\kappa_-(\alpha)}$, $\Theta_{\kappa_+(\alpha)}$ are given, and we define

$$\begin{aligned} \Theta_\alpha(x;x') &= B_{1,2}\big((U^{(1)}(t_\alpha - t_{\kappa_-(\alpha)})\Theta_{\kappa_-(\alpha)}) \\ &\qquad \otimes (U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)})\big)(x;x') \\ &= (U^{(1)}(t_\alpha - t_{\kappa_-(\alpha)})\Theta_{\kappa_-(\alpha)})(x;x') \\ &\quad \cdot \big[(U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)})(x;x) \\ &\quad - (U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)})(x';x')\big]. \end{aligned} \tag{6.29}$$

Clearly, if $\kappa_\pm(\alpha)$ corresponds to a regular leaf vertex, then

$$\Theta_{\kappa_\pm(\alpha)}(x;x') = \phi(x)\overline{\phi(x')} \tag{6.30}$$

and $t_{\ell_{j,\kappa_\pm(\alpha)}} = t_r$. If $\kappa_\pm(\alpha) = m_j$ is the distinguished vertex, we use (6.27).

We iterate this procedure until we obtain the kernel Θ_1 at $\alpha = 1$, which is the unique child vertex of the root vertex.

6.3 Factorization Structure of Kernels

We will now determine the structure of Θ_α.

LEMMA 6.1. *Let $\alpha \in \{1,\dots,m_j\}$. Then every kernel Θ_α can be written as a sum of differences of factorized kernels,*

$$\Theta_\alpha(x;x') = \sum_{\beta_\alpha} c^\alpha_{\beta_\alpha}\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}(x')} \tag{6.31}$$

with at most $2^{m_j-\alpha}$ nonzero coefficients $c^\alpha_{\beta_\alpha} \in \{1,-1\}$.

PROOF. The kernels at the leaf vertices (6.25) have the form (6.31). If $\alpha = m_j$ is the distinguished vertex, Θ_{m_j} is given by (6.27) and evidently has the form (6.31). For the induction step, let us assume that given $\alpha \in \{1, \dots, m_j\}$, the kernels $\Theta_{\alpha'}$ have the form (6.31) for all $\alpha' > \alpha$. Thus, in particular, $\Theta_{\kappa_+(\alpha)}$, $\Theta_{\kappa_-(\alpha)}$ have this form. Then, from (6.29), we find that

$$
\begin{aligned}
\Theta_\alpha(x;x') = \sum_{\substack{\beta_{\kappa_-(\alpha)},\\ \beta_{\kappa_+(\alpha)}}} & c^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}} c^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}} \big(U_{\alpha;\kappa_-(\alpha)} \chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\big)(x) \\
& \cdot \overline{\big(U_{\alpha;\kappa_-(\alpha)} \psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\big)(x')} \\
& \cdot \Big[\big(U_{\alpha;\kappa_+(\alpha)} \chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\big)(x) \overline{\big(U_{\alpha;\kappa_+(\alpha)} \psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\big)(x)} \\
& \quad - \big(U_{\alpha;\kappa_+(\alpha)} \chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\big)(x') \overline{\big(U_{\alpha;\kappa_+(\alpha)} \psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\big)(x')}\Big],
\end{aligned}
\tag{6.32}
$$

where for brevity, we write

$$
U_{\alpha;\alpha'} := e^{i(t_{\ell_{j,\alpha}} - t_{\ell_{j,\alpha'}})\Delta_x}, \quad \alpha, \alpha' \in \{0, 1, \dots, m_j\}
\tag{6.33}
$$

(with $t_0 := t$) for free one-particle propagators. We write the sum on the right-hand side in an arbitrary but fixed order and use the individual terms as definitions for the terms $c^\alpha_{\beta_\alpha}$, $\chi^\alpha_{\beta_\alpha}$, $\psi^\alpha_{\beta_\alpha}$ in

$$
\Theta_\alpha(x;x') = \sum_{\beta_\alpha} c^\alpha_{\beta_\alpha} \chi^\alpha_{\beta_\alpha}(x) \overline{\psi^\alpha_{\beta_\alpha}(x')},
\tag{6.34}
$$

which is of the form (6.31). This iteration terminates when we reach $\alpha = 1$. □

In particular, we have that

$$
J^1_j(\sigma_j; t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}}) = U^{(1)}(t - t_{\ell_{j,1}})\Theta_1.
\tag{6.35}
$$

For convenience, we will notationally suppress the dependence of the functions $\chi^\alpha_{\beta_\alpha}$, $\psi^\alpha_{\beta_\alpha}$ on the time variables $t_{\ell_{\alpha'}}$, but we note that they do not depend on any $t_{\ell_{\alpha'}}$ with $\alpha' < \alpha$.

DEFINITION 6.2. From here on, we reinterpret $\widetilde{\psi}$ in Θ_{m_j} (see (6.27)) as a function that is *independent* of ϕ, $\overline{\phi}$. Only at the end of our analysis, we will substitute $\widetilde{\psi} := |\phi|^2\phi$. For each $\alpha = 1, \dots, m_j$, we call a factor $\chi^\alpha_{\beta_\alpha}$, $\psi^\alpha_{\beta_\alpha}$ in the expansion (6.31) *distinguished* if it is a function of $\widetilde{\psi}$.

Next, we derive recursive bounds on the functions $\chi^\alpha_{\beta_\alpha}$, $\psi^\alpha_{\beta_\alpha}$.

6.4 Key Properties of the Kernels Θ_α

We make the following key observations, which will be crucial for the next steps of our proof.

- The only dependence of Θ_α on the time variable $t_{\ell_{j,\alpha}}$ is via the propagators

$$U_{\alpha;\kappa_\pm(\alpha)} = e^{i(t_{\ell_{j,\alpha}} - t_{\ell_{j,\kappa_\pm(\alpha)}})\Delta} \tag{6.36}$$

  appearing on the right-hand side of (6.32). The kernels $\Theta_{\kappa_\pm(\alpha)}$ at the two child vertices $\kappa_\pm(\alpha)$ of α do not depend on $t_{\ell_{j,\alpha}}$. This will be crucial for the application of Strichartz estimates below.
- The product $\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}(x')}$ in (6.34) either has the form

$$\begin{aligned}\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}}(x') &= \big(U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\big)(x)\overline{\big(U_{\alpha;\kappa_-(\alpha)}\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\big)}(x') \\ &\quad \cdot (U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})(x)\overline{(U_{\alpha;\kappa_+(\alpha)}\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})}(x)\end{aligned} \tag{6.37}$$

  or

$$\begin{aligned}\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}}(x') &= \big(U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\big)(x)\overline{\big(U_{\alpha;\kappa_-(\alpha)}\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\big)}(x') \\ &\quad \cdot \big(U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\big)(x')\overline{\big(U_{\alpha;\kappa_+(\alpha)}\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\big)}(x')\end{aligned} \tag{6.38}$$

  for some values of $\beta_{\kappa_-(\alpha)}$, $\beta_{\kappa_+(\alpha)}$ that depend on β_α. Comparing the left- and right-hand sides, the function $\chi^\alpha_{\beta_\alpha}$ either has the cubic form

$$\begin{aligned}\chi^\alpha_{\beta_\alpha}(x) &= \big(U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\big)(x) \\ &\quad \cdot \big(U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\big)(x)\overline{\big(U_{\alpha;\kappa_+(\alpha)}\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\big)}(x),\end{aligned} \tag{6.39}$$

  or the linear form

$$\chi^\alpha_{\beta_\alpha}(x) = \big(U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\big)(x). \tag{6.40}$$

  Accordingly, $\psi^\alpha_{\beta_\alpha}$ has either linear or cubic form, respectively. It is important that the product $\chi^\alpha_{\beta_\alpha}\overline{\psi^\alpha_{\beta_\alpha}}$ always be of quartic form (6.37) or (6.38). This fact will again be crucial for the application of Strichartz estimates below.
- In the product on the right-hand side of (6.37) (respectively (6.38)), at most one of the four factors is distinguished (see Definition (6.2)). This follows straightforwardly from an induction along decreasing values of α from using the fact that the statement is true for all regular leaf vertices and for the distinguished vertex (6.27).

We may therefore make the following assumption, which leads to notational simplifications but no loss of generality.

Hypothesis 1. In all that follows, we assume for notational convenience that only the functions $\psi^1_{\beta_1}$ and, recursively,

$$\left(\psi^{\kappa^q_+(1)}_{\beta_{\kappa^q_+(1)}}\right)_{q=1}^{Q}$$

are distinguished (i.e., are a function of $\widetilde{\psi}$ in (6.27)). Here

$$\kappa^q_+(1) := \underbrace{\kappa_+(\kappa_+(\cdots(\kappa_+}_{q \text{ times}}(1))\cdots)) \tag{6.41}$$

is the q^{th} iterate of the index $\alpha = 1$ under κ_+, and Q is the number of edges linking $\alpha = 1$ to the distinguished vertex with label $\alpha = m_j$. This is one special case, but all cases can be treated in the same way.

7 Recursive L^2- and H^1-Bounds for the Distinguished Tree

From here on, we abbreviate the notation by writing t_α for $t_{\ell_{j,\alpha}}$ and by referring to the vertex $v_{j,\alpha}$ by its label α. Also, we will say that the time variable t_α is attached to the vertex α.

We have that

$$\begin{aligned}
&\int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \operatorname{Tr}\left(\left|J^1_j(\sigma_j; t, t_1, \dots, t_{m_j})\right|\right) \\
&\quad = \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \operatorname{Tr}(|U^{(1)}(t-t_1)\Theta_1|) \\
&\quad \le \sum_{\beta_1} \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|\psi^1_{\beta_1}\|_{L^2} \|\chi^1_{\beta_1}\|_{L^2}.
\end{aligned} \tag{7.1}$$

We will estimate the last term on the right-hand side based on the recursion formula (6.32) using recursive bounds adapted to a hierarchy of subtrees of τ_j.

Our main goal is to propagate the L^2-norm in (7.1) along edges of τ_j that connect the vertex $\alpha = 1$ to the distinguished vertex $\alpha = m_j$ in order to obtain a bound

$$\|\widetilde{\psi}\|_{L^2} = \||\phi|^2\phi\|_{L^2} \le C\|\phi\|^3_{H^1} \tag{7.2}$$

that can be controlled by the growth condition (4.18).

7.1 Recursive Bounds

We let $\tau_{j,\alpha}$ denote the subtree of τ_j with root at the vertex labeled by α. Moreover, we denote by

$$\int \Big[\prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'}\Big] \equiv \int_{[0,T)^{d_\alpha}} \Big[\prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'}\Big] \tag{7.3}$$

integration with respect to all time variables attached to the internal and root vertices of the subtree $\tau_{j,\alpha}$ with root at α. Here, d_α denotes the total number of internal and root vertices of $\tau_{j,\alpha}$.

LEMMA 7.1. *Let $\kappa_-(\alpha)$ and $\kappa_+(\alpha)$ label the two child vertices of the vertex labeled by α. Assume that either* (6.37) *or* (6.38) *is given. Then the following recursive bounds hold*:

- *Bound on L^2-level.*

$$\begin{aligned}
&\int \Big[\prod_{\alpha'\in\tau_{j,\alpha}} dt_{\alpha'}\Big] \|\psi^{\alpha}_{\beta_\alpha}\|_{L^2} \|\chi^{\alpha}_{\beta_\alpha}\|_{H^1} \\
&\quad\le CT^{\frac{1}{2}} \int \Big[\prod_{\alpha'\in\tau_{j,\kappa_-(\alpha)}} dt_{\alpha'}\Big] \|\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{H^1} \|\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{H^1} \\
&\qquad\cdot \int \Big[\prod_{\alpha'\in\tau_{j,\kappa_+(\alpha)}} dt_{\alpha'}\Big] \|\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{L^2} \|\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{H^1}.
\end{aligned} \tag{7.4}$$

- *Bound on H^1-level.*

$$\begin{aligned}
&\int \Big[\prod_{\alpha'\in\tau_{j,\alpha}} dt_{\alpha'}\Big] \|\psi^{\alpha}_{\beta_\alpha}\|_{H^1} \|\chi^{\alpha}_{\beta_\alpha}\|_{H^1} \\
&\quad\le CT^{\frac{1}{2}} \int \Big[\prod_{\alpha'\in\tau_{j,\kappa_-(\alpha)}} dt_{\alpha'}\Big] \|\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{H^1} \|\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{H^1} \\
&\qquad\cdot \int \Big[\prod_{\alpha'\in\tau_{j,\kappa_+(\alpha)}} dt_{\alpha'}\Big] \|\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{H^1} \|\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{H^1}.
\end{aligned} \tag{7.5}$$

PROOF. This can be inferred from the bounds (6.18) and (6.20) as follows:

- *Bound on L^2-level.* By applying (6.18) to (6.39) and (6.40) with respect to the time variable t_α, and recalling that

$$U_{\alpha;\kappa_-(\alpha)} = e^{i(t_\alpha - t_{\kappa_-(\alpha)})\Delta}, \tag{7.6}$$

we obtain

$$\begin{aligned}
&\int_{[0,T)^{d_\alpha}} \Big[\prod_{\alpha'\in\tau_{j,\alpha}} dt_{\alpha'}\Big] \|\psi^{\alpha}_{\beta_\alpha}\|_{L^2} \|\chi^{\alpha}_{\beta_\alpha}\|_{H^1} \\
&\quad\le CT^{1/2} \int_{[0,T)^{d_\alpha - 1}} \Big[\prod_{\alpha'\in\tau_{j,\kappa_-(\alpha)}\cup\tau_{j,\kappa_+(\alpha)}} dt_{\alpha'}\Big] \big\|e^{-it_{\kappa_-(\alpha)}\Delta} \chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\big\|_{H^1} \\
&\qquad\cdot \Big\|e^{-it_{\kappa_-(\alpha)}\Delta} \psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\Big\|_{H^1} \Big\|e^{-it_{\kappa_+(\alpha)}\Delta} \chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\Big\|_{H^1} \\
&\qquad\cdot \Big\|e^{-it_{\kappa_+(\alpha)}\Delta} \psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\Big\|_{L^2} =
\end{aligned} \tag{7.7}$$

$$= CT^{1/2} \int_{[0,T)^{d_{\kappa_-(\alpha)}}} \Big[\prod_{\alpha' \in \tau_{j,\kappa_-(\alpha)}} dt_{\alpha'} \Big] \| \chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}} \|_{H^1} \| \psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}} \|_{H^1}$$
$$\cdot \int_{[0,T)^{d_{\kappa_+(\alpha)}}} \Big[\prod_{\alpha' \in \tau_{j,\kappa_+(\alpha)}} dt_{\alpha'} \Big] \| \chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}} \|_{H^1} \| \psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}} \|_{L^2}.$$

Here we first used Cauchy-Schwarz in the t_α-integral. In the last step, we used that $\psi^\alpha_{\beta_\alpha}$, $\chi^\alpha_{\beta_\alpha}$ depend only on the time variables $t_{\alpha'}$ that are attached to the vertices of the subtree $\tau_{j,\alpha}$ rooted at the vertex α for every $\alpha \in \{1, \dots, m_j - 1\}$. Moreover, we used that $e^{-it_{\kappa_\pm(\alpha)}\Delta}$ are unitary in L^2 and H^1.

- *Bound on H^1-level.* Using (6.20), we obtain

$$\int_{[0,T)^{d_\alpha}} \Big[\prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \Big] \| \psi^\alpha_{\beta_\alpha} \|_{H^1} \| \chi^\alpha_{\beta_\alpha} \|_{H^1}$$
$$\le CT^{1/2} \int_{[0,T)^{d_\alpha - 1}} \Big[\prod_{\alpha' \in \tau_{j,\kappa_-(\alpha)} \cup \tau_{j,\kappa_+(\alpha)}} dt_{\alpha'} \Big] \| e^{-it_{\kappa_-(\alpha)}\Delta} \chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}} \|_{H^1}$$
$$\cdot \; \| e^{-it_{\kappa_-(\alpha)}\Delta} \psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}} \|_{H^1} \| e^{-it_{\kappa_+(\alpha)}\Delta} \chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}} \|_{H^1}$$
$$\cdot \; \| e^{-it_{\kappa_+(\alpha)}\Delta} \psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}} \|_{H^1} \tag{7.8}$$
$$= CT^{1/2} \int_{[0,T)^{d_{\kappa_-(\alpha)}}} \Big[\prod_{\alpha' \in \tau_{j,\kappa_-(\alpha)}} dt_{\alpha'} \Big] \| \chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}} \|_{H^1} \| \psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}} \|_{H^1}$$
$$\cdot \int_{[0,T)^{d_{\kappa_+(\alpha)}}} \Big[\prod_{\alpha' \in \tau_{j,\kappa_+(\alpha)}} dt_{\alpha'} \Big] \| \chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}} \|_{H^1} \| \psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}} \|_{H^1},$$

by proceeding as above for the bounds on the L^2-level. □

8 Concluding the Proof

Using Lemma 7.1, we can now prove the following main estimates for the distinguished tree in Proposition 8.1 and for regular trees in Proposition 8.2.

PROPOSITION 8.1. *Assume that τ_j is the distinguished tree. Then, the bound*

$$\int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \operatorname{Tr}\left(\left|J_j^1(\sigma_j; t, t_1, \dots, t_{m_j})\right|\right) \leq 2^{m_j} C^{m_j} T^{(m_j-1)/2} \|\phi\|_{H_1}^{m_j+1} \tag{8.1}$$

holds.

PROOF. To begin with,

$$\begin{aligned}
&\int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \operatorname{Tr}\left(\left|J_j^1(\sigma_j; t, t_{\ell_1}, \dots, t_{m_j})\right|\right) \\
&\quad = \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \operatorname{Tr}(|U^{(1)}(t-t_1)\Theta_1|) \\
&\quad \leq \sum_{\beta_1} \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|\psi_{\beta_1}^1\|_{L^2} \|\chi_{\beta_1}^1\|_{L^2} \\
&\quad \leq \sum_{\beta_{\kappa_-(1)}, \beta_{\kappa_+(1)}} C T^{1/2} \int \Big[\prod_{\alpha' \in \tau_{j,\kappa_-(1)}} dt_{\alpha'}\Big] \|\psi_{\beta_{\kappa_-(1)}}^{\kappa_-(1)}\|_{H^1} \|\chi_{\beta_{\kappa_-(1)}}^{\kappa_-(1)}\|_{H^1} \qquad (8.2)\\
&\quad\qquad \cdot \int \Big[\prod_{\alpha' \in \tau_{j,\kappa_+(1)}} dt_{\alpha'}\Big] \|\psi_{\beta_{\kappa_+(1)}}^{\kappa_+(1)}\|_{L^2} \|\chi_{\beta_{\kappa_+(1)}}^{\kappa_+(1)}\|_{H^1}, \qquad (8.3)
\end{aligned}$$

where we first used (6.35), then (6.34), and subsequently (7.4) with respect to the integral in t_1, at the vertex $\alpha = 1$. Then we used (6.32) and the fact that $|c_{\beta_{\kappa_i(1)}}^{\kappa_i(1)}| = 1$. By Hypothesis 1, $\psi_{\beta_{\kappa_+(1)}}^{\kappa_+(1)}$ is the only function on the last two lines that is distinguished, which is why it is the only term bounded in L^2.

We first bound the integral (8.2). To this end, we iterate the bound (7.5) on the H^1-level until we reach all leaf vertices of the subtree $\tau_{j,\kappa_-(1)}$. It follows from Hypothesis 1 that $\tau_{j,\kappa_-(1)}$ does not contain the distinguished vertex; therefore all leaf vertices of $\tau_{j,\kappa_-(1)}$ are regular. Then, we find that

$$\int \Big[\prod_{\alpha' \in \tau_{j,\kappa_-(1)}} dt_{\alpha'}\Big] \|\psi_{\beta_{\kappa_-(1)}}^{\kappa_-(1)}\|_{H^1} \|\chi_{\beta_{\kappa_-(1)}}^{\kappa_-(1)}\|_{H^1} \leq C T^{d_{\kappa_-(1)}/2} \|\phi\|_{H^1}^{2b_{\kappa_-(1)}} \tag{8.4}$$

where b_α is the number of regular leaf vertices of the subtree $\tau_{j,\alpha}$ rooted at α, and d_α is the number of internal vertices of the subtree $\tau_{j,\alpha}$.

Next, we bound the integral (8.3). To this end, we iterate both the bound (7.4) on the L^2-level and the bound (7.5) on the H^1-level until we reach all leaf vertices of the subtree $\tau_{j,\kappa_+(1)}$, including the distinguished vertex v_{j,m_j}. The L^2-norm is in every step applied to the term $\psi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}$, which is the only function in (6.37), respectively (6.38), which is distinguished, due to Hypothesis 1. The iteration terminates when all regular leaf vertices and the distinguished vertex are reached. We then obtain that

$$
\begin{aligned}
&\int \Big[\prod_{\alpha' \in \tau_{j,\kappa_+(1)}} dt_{\alpha'} \Big] \| \psi^{\kappa_+(1)}_{\beta_{\kappa_+(1)}} \|_{L^2} \| \chi^{\kappa_+(1)}_{\beta_{\kappa_+(1)}} \|_{H^1} \\
&\quad \le C^{m_j} T^{(d_{\kappa_+(1)}-1)/2} \|\phi\|_{H^1}^{2b_{\kappa_+(1)}} \|\widetilde{\psi}\|_{L^2} \\
&\quad \le C^{m_j} T^{(d_{\kappa_+(1)}-1)/2} \|\phi\|_{H^1}^{2b_{\kappa_+(1)}+3}
\end{aligned}
\tag{8.5}
$$

where the L^2-norm has been moved to the distinguished vertex, hence the factor $\|\widetilde{\psi}\|_{L^2}$. At this point, we substituted $\widetilde{\psi} := |\phi|^2\phi$ and used the Sobolev embedding.

Combining the bounds (8.4) and (8.5), we obtain (8.1), where all leaves contribute a factor $\|\phi\|_{H^1}^2$. The factor 2^{m_j} bounds the number of terms in the sum over $\beta_{\kappa_-(1)}, \beta_{\kappa_+(1)}$ in (8.2). □

Similarly, we find for regular trees the following:

PROPOSITION 8.2. *Assume that τ_j is a regular tree. Then, the bound*

$$
\int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \operatorname{Tr}\big(\big| J_j^1(\sigma_j; t, t_1, \dots, t_{m_j}) \big| \big) \le 2^{m_j} C^{m_j} T^{m_j/2} \|\phi\|_{H_1}^{m_j+1}
\tag{8.6}
$$

holds.

PROOF. For a regular tree, we have

$$
\begin{aligned}
J_j^1(\sigma_j; t, t_1, \dots, t_{m_j}) = {} & U^{(1)}(t - t_1) B_{\sigma_j(2),2} \\
& \cdots U^{(m_j)}(t_{m_j-1} - t_{m_j}) B_{\sigma_j(m_j+1), m_j+1} \\
& \cdot U^{(m_j+1)}(t_{m_j}) \big(|\phi\rangle\langle\phi| \big)^{\otimes (m_j+1)}.
\end{aligned}
\tag{8.7}
$$

The key difference between (8.7) and the corresponding expression (6.3) for the distinguished tree is the presence of the free propagator $U^{(m_j+1)}(t_{m_j})$ on the last line. The proof is immediately obtained from the proof of Proposition 8.1, by using

$$
\begin{aligned}
&\int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \operatorname{Tr}\big(\big| J_j^1(\sigma_j; t, t_1, \dots, t_{m_j}) \big| \big) \\
&\quad = \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \operatorname{Tr}(|U^{(1)}(t - t_1)\Theta_1|) \le
\end{aligned}
\tag{8.8}
$$

$$\leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|\psi^1_{\beta_1}\|_{L^2} \|\chi^1_{\beta_1}\|_{L^2}$$
$$\leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|\psi^1_{\beta_1}\|_{H^1} \|\chi^1_{\beta_1}\|_{H^1},$$

and by iterating the bound (7.5) on the H^1-level until we reach all leaf vertices of τ_j. Because all leaves of τ_j are regular, no L^2-level bound is necessary. □

Going back to (4.27), we find from

$$J^k(\sigma; t, t_1, \dots, t_r; \underline{x}_k; \underline{x}'_k) = \prod_{j=1}^{k} J^1_j(\sigma_j; t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}}; x_j; x'_j) \tag{8.9}$$

that

$$\begin{aligned} &\int_{[0,t]^{r-1}} dt_1 \cdots dt_{r-1} \operatorname{Tr}(|J^k(\sigma; t, t_1, \dots, t_r)|) \\ &\quad = \int_{[0,t]^{r-1}} dt_1 \cdots dt_{r-1} \prod_{j=1}^{k} \operatorname{Tr}\left(\left|J^1_j(\sigma_j; t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}})\right|\right) \\ &\quad \leq 2^r C^r T^{(r-1)/2} \|\phi\|_{H_1}^{2(k+r)}, \end{aligned} \tag{8.10}$$

by combining the estimates from the $k-1$ regular trees and from the distinguished tree. The factor 2^r is obtained from the product of factors 2^{m_j} from all trees τ_j, both regular and distinguished.

Then we observe that for $t \in [0, T)$,

$$\begin{aligned} \operatorname{Tr}(|\gamma^{(k)}(t)|) &\leq (2CT)^{(r-1)/2} \sum_{i=1,2} \int_0^T dt_r \int d\mu^{(i)}_{t_r}(\phi) \|\phi\|_{H^1}^{2(r+k)} \\ &\leq (2CT)^{(r+1)/2} \sup_{t_r \in [0,T)} \sum_{i=1,2} \int d\mu^{(i)}_{t_r}(\phi) \|\phi\|_{H^1}^{2(r+k)}. \end{aligned} \tag{8.11}$$

The growth condition (4.18) implies that this is bounded by

$$(8.11) < 2M^{2k-2}(2CM^4T)^{(r+1)/2} \longrightarrow 0, \quad r \to \infty, \tag{8.12}$$

for $T < (2CM^4)^{-1}$ (which is in particular uniform in k). Since k is fixed and r is arbitrary, we conclude that

$$\operatorname{Tr}(|\gamma^{(k)}(t)|) = 0, \quad t \in [0, T), \tag{8.13}$$

which implies that $\gamma^{(k)}(t) = 0$ for $t \in [0, T)$, and hence uniqueness holds.

Moreover, it can be easily checked that

$$\gamma^{(k)}(t) = \int d\mu(\phi)(|S_t(\phi)\rangle\langle S_t(\phi)|)^{\otimes k} \quad \forall k \in \mathbb{N}, \tag{8.14}$$

is a mild solution of the GP hierarchy in $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ with initial data

$$\gamma^{(k)}(0) = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad \forall k \in \mathbb{N}. \tag{8.15}$$

By uniqueness, it is the only such solution. This proves Theorem 2.3. □

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