DYADIC MODELS FOR THE EQUATIONS OF FLUID MOTION

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1. Introduction

In this note we describe results addressing the behavior of a certain toy model for the equations of fluid motion, the so called "dyadic" model. Some of the results that will be reviewed here were obtained in last few years in collaborations with Alexey Cheskidov and Nets Katz. The authors are thankful to their coauthors for inspiring collaborations. The review presented here is far from being exhaustive and it is beyond the scope of this short paper to describe all the large body of work on the discretized models for the fluid equations.

1.1. **Equations of fluid motion.**

The partial differential equations that describe the most crucial properties of the motion of an incompressible, inviscid fluid are the Euler equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

with the initial condition

$$u(x, 0) = u_0(x), \tag{1.3}$$

for the unknown velocity vector field $u = u(x,t) \in \mathbb{R}^d$ and the pressure $p = p(x,t) \in \mathbb{R}$, where $x \in \mathbb{R}^d$ and $t \in [0, \infty)$. They are derived for an incompressible, inviscid fluid with constant density. Despite the fact that Euler introduced them almost two and a half centuries ago, some basic questions concerning Euler equations in 3 dimensions are still unsolved. For example, it is an outstanding problem of fluid dynamics to find out if solutions of the 3D Euler equations form singularities in finite time.

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The equations that reflect the most fundamental properties of viscous, incompressible fluids are the Navier-Stokes equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \tag{1.4}$$

$$\nabla \cdot u = 0, \tag{1.5}$$

and the initial condition

$$u(x,0) = u_0(x), \tag{1.6}$$

and appropriate boundary conditions. As with the Euler equations the theory of the Navier-Stokes equations in three dimensions is far from being complete. The major open problems are global existence, uniqueness and regularity of smooth solutions of the Navier-Stokes equations in 3D. For the precise formulation of this open problem see [10]. One way of looking into the problem is via weak solutions that were introduced in the context of the Navier-Stokes equations by Leray [24] - [26] in 1930s. Leray [26] and Hopf [13] showed existence of a global weak solution of the Navier-Stokes equations. However questions addressing uniqueness and regularity of these solutions have not been answered yet. Important contributions in understanding partial regularity and conditional uniqueness of weak solutions have been made by many authors, including, Ladyzhenskaya [19], Prodi [37], Serrin [42], Scheffer [38] - [40], Caffarelli-Kohn-Nirenberg [4], Lin [27] and Escauriaza-Seregin-Šverak [9]. Another approach in studying behavior of the Navier-Stokes equations is to construct solutions via a fixed point theorem. In the context of the Navier-Stokes equations this approach was pioneered by Kato and Fujita [14] and continued by many authors including the result of Koch and Tataru [18]. However the existence of such solutions to the Navier-Stokes equations has been proved only locally in time and globally for small initial data.

Now we recall an important conserved quantity of the fluid equations. As a consequence of skew-symmetry property of the nonlinear term

$$\langle (u \cdot \nabla)u, u \rangle_{L^2(\mathbb{R}^3)} = 0$$

and divergence free condition, the classical solutions to the Euler equations (1.1)-(1.3) satisfy conservation of energy:

$$||u(\cdot,T)||^2_{L^2} = ||u_0||^2_{L^2},$$

while classical solutions to the Navier-Stokes equations (1.4)-(1.6) satisfy decay of energy:

$$||u(\cdot,T)||^2_{L^2} = ||u_0||^2_{L^2} - 2\int_0^T \nu \langle (-\Delta)u, u \rangle.$$

1.2. **A dyadic model.** Many results concerning Euler and Navier-Stokes equations in 3 space dimensions use two important properties of these equations: conservation (decay) of energy and scaling property of the nonlinear term, see, for example, [2], [4]. In this article we review results concerning a dyadic model of fluid equations that shares[1] these two properties with actual Euler and Navier-Stokes equations in 3D. The precise formulation of the model is in Section 2 of the paper.

Similar discretized models have been constructed and analyzed with a goal to capture important properties of fluid equations. These models belong to a class of "shell models" which, in general, simulate the energy cascade among the set of velocities u_n, where u_n stands for the velocity associated to the n^{th} shell. In all these models the nonlinearity of the Euler equations $(u \cdot \nabla)u$ is drastically simplified: only local neighboring interactions between certain scales are considered. Depending on the simplification of the nonlinear term the models differ in the number of conserved quantities and in presence of a certain "monotonicity" property that will be discussed in Section 3 of this paper. Among the first examples of a discretized model is the one introduced by Gledzer [12] which was later generalized by Ohkitani and Yamada [35] and is now known as the GOY model. There are a number of other types of discretized models that have been studied recently: for example, the model of Constantin et al [7] for which the authors prove global regularity and the existence of a finite dimensional global attractor. The book of Bohr et al [3] offers a survey of various results concerning shell models.

The dyadic model that we discuss in the present paper was introduced by Katz, Pavlović in [15] as a test model for their study of partial regularity for solutions to the 3D Navier-Stokes equations with hyper-dissipation. However the model was investigated further. Local in time existence of solutions to dyadic Euler equations was obtained in [11]. Finite time blow-up was initially proved for dyadic Euler equations in [15] by exploiting conservation of energy and a certain *monotonicity* present in the model. Such a monotonicity property resembles monotonicity of certain quantities present in so called "cooperative" systems (see for example the work of Palais [36] and the work of Bernoff and Bertozzi [1] where singularities in a modified Kuramoto-Sivashinsky equation were identified). Finite time blow-up was investigated further by Waleffe [46] and sharpened by Kiselev, Zlatoš [30]. The famous question of global existence of solutions to the Navier-Stokes equations was answered in the dyadic context by Nazarov [34]. More precisely, the solution to the dyadic Navier-Stokes equations stays bounded in a certain C^k space provided that it started in the same C^k space. The main tool in Nazarov's proof is the observation that if the system of ODEs which

[1]We remark that the dyadic model conserves energy, provided that the function which mimics velocity belongs to the Sobolev space H^s with s large enough to justify a certain summation incorporated in the expression (2.6). For details, see [6].

describes the dyadic Navier-Stokes equations is truncated, it will scatter all energy. A 3 dimensional vector model for the incompressible Euler equations was introduced in Friedlander, Pavlović [11], which is similar in some features to a discretized approximate model constructed by Dinaburg and Sinai [8] for the Navier-Stokes equations in Fourier space. It was shown in [11] that for special initial data the evolution equations of the divergence free vector model reduced to the scalar dyadic Euler system and finite time blow-up occurs in this model for the 3 dimensional incompressible Euler equations. We remark that finite time blow-up was exhibited even for a version of the dyadic model which allows certain "small" degrees of dissipation, see [15]. Recent work on finite time blow-up for dissipative dyadic models was performed by Cheskidov in [5]. Also recently we have proved [6] that a version of the inviscid dyadic model with forcing has a unique equilibrium which is spectrally and nonlinearly stable. For a critical s_0, finite time blow-up occurs in H^s for $s \geq s_0$. In H^s, $s < s_0$, the energy actually decays even though there is no viscosity in the model. Such a phenomenon of "anomalous dissipation" is also observed in a linear discretized model by Mattingly et al [33].

The above mentioned results can be divided in two categories: the results which are obtained just for the model thanks to the drastic simplifications that are incorporated in the model and the results obtained for the model that can be generalized to the actual equations. In Section 3 of the present paper we recall a finite time blow-up for the dyadic Euler equations [16] which belongs to the first category, while in Section 4 we describe a partial regularity result for the dyadic Navier-Stokes equations with hyper-dissipation [15] which was generalized to the result for actual equations.

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2. Dyadic model

The dyadic model is introduced in the following way. Throughout the paper we shall use the definition of a dyadic cube that say that a cube Q in $\mathbb{R}^3$ is called a dyadic cube if its sidelength is integer power of 2, 2^l, and the corners of the cube are on the lattice $2^l\mathbb{Z}^3$. Let $\mathcal{D}$ denote the set of dyadic cubes in $\mathbb{R}^3$. Let $\mathcal{D}_j$ denote the subset of dyadic cubes having sidelength 2^{-j}. We define PQ, the parent of Q, to be the unique dyadic cube in $\mathcal{D}_{j(Q)-1}$ which contains Q. We define $\mathcal{C}^k(Q)$, the kth order grandchildren of Q to be the set of those cubes in $\mathcal{D}_{j(Q)+k}$ which are contained in Q.

The first simplification comes by replacing vector valued function u by a scalar valued one. We denote an orthonormal family of wavelets by $\{w_Q\}$, with w_Q the wavelet associated to the spatial dyadic cube $Q \in \mathcal{D}_j$. Then u can be represented as:

$$u(x,t) = \sum_Q u_Q(t) w_Q(x).$$

Note that due to spatial localization of w_Q

$$||w_Q||_{L^\infty} \sim 2^{\frac{3j(Q)}{2}}. \tag{2.1}$$

On the other hand

$$||\nabla w_Q||_{L^2} \sim 2^j. \tag{2.2}$$

Having in mind (2.1) and (2.2) we define a cascade down operator through its Q^{th} coefficient as follows:

$$(C_d(u,v))_Q = 2^{\frac{5j(Q)}{2}} u_{PQ} v_{PQ}.$$

Similarly we define a cascade up operator as the adjoint of $C_d(u,v)$ via:

$$(C_u(u,v))_Q = 2^{\frac{5(j(Q)+1)}{2}} u_Q \sum_{Q' \in C^1(Q)} v_{Q'}.$$

Now we define the cascade operator

$$C(u,v) = C_u(u,v) - C_d(u,v).$$

Having defined Laplacian as $\Delta(w_Q) = 2^{2j} w_Q$, we introduce the following model equations:

- Dyadic Euler equation:

$$\frac{du}{dt} + C(u,u) = 0. \tag{2.3}$$

- Dyadic Navier-Stokes equation:

$$\frac{du}{dt} + C(u,u) + \Delta u = 0. \tag{2.4}$$

- Dyadic Navier-Stokes equation with hyper-dissipation:

$$\frac{du}{dt} + C(u,u) + (\Delta)^\alpha u = 0. \tag{2.5}$$

By construction of cascade operators,

$$\langle C_u(u,u),u\rangle = \langle C_d(u,u),u\rangle,$$

which implies

$$\langle C(u,u),u\rangle = 0. \tag{2.6}$$

A simple consequence of (2.6) is conservation of energy for the dyadic Euler equations and decay of energy for the dyadic Navier-Stokes equations. Hence the model shares conserved quantities with fluid equations.

3. A "dream" result proved only in the dyadic context

In this section we briefly describe finite time blow-up obtained for solutions to dyadic Euler equations in [16]. Such a result can be thought of as a result obtained for the model equations themselves thanks to certain monotonicity property specific for the dyadic Euler equation (2.3) which combined with the conservation of energy implies cascade of energy into higher and higher frequency scales.

Let $||u||_{L^2}$ denote the L^2 norm of u. For any $s > 0$, we define

$$||u||_{H^{2s}} = ||u||_{L^2} + ||(\Delta)^s u||_{L^2}.$$

We consider functions u that satisfy the dyadic Euler equations (2.3) and all of whose coefficients u_Q are initially positive. After writing Duhamel's formula

$$u_Q(t) = \frac{1}{\mu(t)}(u_Q(0) + \int_0^t 2^{\frac{5j(Q)}{2}} u_{PQ}^2(\tau)\mu(\tau)d\tau), \tag{3.1}$$

where

$$\mu(t) = e^{\int_0^t 2^{\frac{5(j(Q)+1)}{2}} \sum_{Q'\in C^1(Q)} u_{Q'}(\tau)d\tau},$$

it can be seen that this class of functions is preserved by the dyadic Euler flow. We remark that this "positivity" is a specific characteristic of the exact form of the dyadic Euler equations (2.3) and does not hold for the actual Euler equations.

Let E denote the energy $E := \langle u,u\rangle$. The energy can be seen as being divided up among the cubes Q:

$$E = \sum_{Q\in\mathcal{D}} E_Q, \text{ where } E_Q = u_Q^2.$$

A local description of energy flow along the tree $\mathcal{D}$ is given by:

$$\frac{d}{dt}E_Q = E_{Q,in} - E_{Q,out}, \tag{3.2}$$

where

$$E_{Q,in} = 2^{\frac{5j(Q)}{2}} 2u_{PQ}^2 u_Q,$$

and

$$E_{Q,out} = \sum_{Q' \in \mathcal{C}^1(Q)} E_{Q',in}.$$

Hence energy is flowing always from larger cubes to smaller ones and indeed it flows along the edges of the tree $\mathcal{D}$. We can see this precisely after we introduce the extended Carleson box of a cube Q by

$$\mathcal{C}_0(Q) = \bigcup_{k=0}^{\infty} \mathcal{C}^k(Q),$$

and the energy of an extended Carleson box by

$$E_{\mathcal{C}_0(Q)} = \sum_{Q_1 \in \mathcal{C}_0(Q)} E_{Q_1}.$$

Then after writing (3.2) for all cubes in $\mathcal{C}_0(Q)$ it can be shown (see [16] for details):

Proposition 3.1. *Let u be a time-varying function with initially positive coefficients evolving according to the dyadic Euler equations. Then for any Q, the function $E_{\mathcal{C}_0(Q)}$ is monotone increasing in time.*

Now we combine the monotonicity property stated in Proposition 3.1 with conservation of energy to obtain the main iterative result of the form

Lemma 3.2. *Fix j_0 sufficiently large. Then there is a sufficiently small $0 < \epsilon < 1$ so that if at time t_0, we have*

$$E_{\mathcal{C}_0(Q)} \geq 2^{-(3+\epsilon)j(Q)}, \tag{3.3}$$

with $j(Q) > j_0$, then there is some t with $t < t_0 + 2^{-\epsilon j(Q)}$ and a cube $Q' \in \mathcal{C}^1(Q)$ so that at time t, we have $E_{\mathcal{C}_0(Q')} \geq 2^{-(3+\epsilon)j(Q')}$.

As a corollary of the iterative Lemma 3.2 we conclude:

Theorem 3.3. *Let u be a solution to the dyadic Euler equations which has initially all positive coefficients and $E_Q(0) > 2^{-(3+\epsilon)j(Q)}$, for some Q with $j(Q) > j_0$ with j_0 as in the previous lemma. Then the $H^{\frac{3}{2}+\epsilon}$ norm of u becomes unbounded in finite time.*

We sketch the proof of the theorem.

Proof We apply Lemma 3.2. We find a cube Q_1 properly contained in Q and a time $t_1 < 2^{-\epsilon j(Q)}$ so that at t_1 we have $E_{\mathcal{C}_0(Q_1)} > 2^{-(3+\epsilon)j(Q_1)}$.

We iterate this procedure finding a cube Q_k contained in Q_{k-1} and a time t_k so that $t_{k-1} \leq t_k < t_{k-1} + 2^{-\epsilon j(Q_{k-1})}$ and at time t_k we have $E_{\mathcal{C}_0(Q_k)} > 2^{-(3+\epsilon)j(Q_k)}$.

Estimating just using the coefficients of $\mathcal{C}_0(Q_k)$, we see that at time t_k, we have that

$$\begin{aligned}
||u||^2_{H^{\frac{3}{2}+\epsilon}} &\geq \sum_{j \geq j(Q_k)} 2^{(3+2\epsilon)j} \sum_{Q \in \mathcal{D}_j} u_Q^2(t_k) \\
&\geq 2^{(3+2\epsilon)j(Q_k)} \sum_{j \geq j(Q_k)} \sum_{Q \in \mathcal{D}_j} u_Q^2(t_k) \\
&\geq 2^{(3+2\epsilon)j(Q_k)} E_{\mathcal{C}_0(Q_k)}(t_k) \\
&\geq 2^{\epsilon j(Q_k)}.
\end{aligned}$$

Since $j(Q_k)$ is an increasing sequence of integers, this is going to ∞. However

$$t_k = (t_k - t_{k-1}) + (t_{k-1} - t_{k-2}) + \cdots + t_1 \leq 2^{-\epsilon j(Q)} + \sum_{l=1}^{k-1} 2^{-\epsilon j(Q_l)},$$

and the $j(Q_l)$'s are an increasing sequence of integers, we see that the sequence $\{t_k\}$ converges to a finite limit. ■

4. The dyadic model motivates a result for the fluid equations

As a contrast to results such as a finite time blow-up for the Euler equations there are instances where results obtained for dyadic models can be generalized to actual fluid equations. These are results which are based on properties that model equations share with actual equations, i.e. conservation (decay) of energy and scaling property of the nonlinear term. An example of such a result is a partial regularity result for the dyadic Navier-Stokes equations with hyper-dissipation (2.5) obtained in [15].

The Navier-Stokes equations with hyper-dissipation $(-\Delta)^\alpha$ are given by

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = -(-\Delta)^\alpha u, \tag{4.1}$$

where u is a time-dependent divergence free vector field in $\mathbb{R}^3$. One sets the initial condition

$$u(x,0) = u_0(x) \tag{4.2}$$

where $u_0(x) \in C_c^\infty(\mathbb{R}^3)$.

It has been shown [28], [29], [32] that the system (4.1) - (4.2) admits global solutions provided that $\alpha \geq \frac{5}{4}$. On the other hand, when $\alpha = 1$ the equations

(4.1) - (4.2) coincide with the Navier-Stokes equations. Caffarelli, Kohn, and Nirenberg [4] showed that when $\alpha = 1$, the singular set of a generalized weak solution to the system (4.1),(4.2) has parabolic Hausdorff dimension at most 1.

In [15] it has been proved that if T is the time of first breakdown for the system (4.1),(4.2), with $1 < \alpha < \frac{5}{4}$ then the Hausdorff dimension of the singular set at time T is at most $5 - 4\alpha$.

The authors in [15] could not directly generalize the proof of Caffarelli, Kohn, and Nirenberg [4]. More precisely, the approach presented in [4] relies on the "generalized energy inequality" which is based on the following property of the divergence free heat equation:

$$\int \langle (\frac{\partial}{\partial t} - \Delta)u, \phi u\rangle dt = \int (\frac{1}{2}\frac{\partial}{\partial t}(\langle u, \phi u\rangle) + \langle \nabla u, \phi \nabla u\rangle - \langle (\frac{1}{2}\frac{\partial \phi}{\partial t} + \Delta\phi)u, u\rangle)dt,$$

where ϕ is any bump function compactly supported in space and time and u a divergence free vector field. To circumvent the problem of not having the "generalized energy inequality" in [15] techniques of microlocal analysis are used, by localizing in frequency and in space. A "quasi" version of the "generalized energy inequality" is obtained which works for certain neighboring cubes and allows the authors to prove a critical level of regularity outside of "bad" cubes in which, intuitively speaking, "nonlinearity dominates the dissipation term". Then a barrier estimate which guarantees arbitrary regularity in the interior of a cube Q is proved, if the critical regularity is known for cubes containing it and for boundary cubes of the cube Q. On the other hand an estimate on the size of the set of bad cubes is given. Such an estimate on the size of the set of bad cubes has been noticed on the level of the dyadic model first. Thus in this context the dyadic model was used as a test model. In the rest of this section we recall the dyadic heuristic that gives an upper bound on the set of singular points.

First let us recall the definition of Hausdorff dimension and state a lemma which is used as a tool in proving an upper bound on the Hausdorff dimension. Given any set $A \subset \mathbb{R}^n$, the d-dimensional Hausdorff measure of A, $\mathcal{H}^d(A)$, is given by:

$$\mathcal{H}^d(A) = \lim_{\rho \longrightarrow 0} \mathcal{C}^d_\rho(A),$$

where $\mathcal{C}^d_\rho(A)$ is defined as:

$$\mathcal{C}^d_\rho(A) = \inf_{C \in \mathcal{C}_\rho(A)} \sum_{B \in C} r(B)^d,$$

where $r(B)$ is the radius of ball B and $\mathcal{C}_\rho(A)$ denotes the set of all coverings of A by balls of radius less than or equal to ρ. The Hausdorff dimension is given by:

$$\inf_{\mathcal{H}^d(A)=0} d.$$

In order to find an upper bound on the Hausdorff dimensions of the set of singular points the following result is utilized, which intuitively speaking gives an upper bound for the Hausdorff dimension of a set, provided that one is able to discretize the particular set so that at level j it could be seen as collections of not more than 2^{jd} balls of radius 2^{-j}, i.e.

Lemma 4.1. *Let $A_1, \dots, A_j, \dots$ be a sequence of collections of balls in $\mathbb{R}^n$ so that each element of A_j has radius 2^{-j}. Suppose that $\#(A_j) \le 2^{jd}$. Define*

$$A = \limsup_{j \longrightarrow \infty} A_j,$$

to be the set of points in infinitely many of the $\cup_{B \in A_j} B$'s. Then the Hausdorff dimension of A is at most d.

Now we consider dyadic model for the Navier-Stokes equation with hyper-dissipation (2.5). Hausdorff dimension of the set of singular points will be estimated at the first time of blow up, T. In order to do so, notice that the cascade operator $C(u,u)$ on scale j looks roughly like $2^{\frac{5j}{2}} u_Q^2$, while the dissipation term gives decay like $2^{2\alpha j} u_Q$. Hence as long as $u_Q < 2^{-\frac{j}{2}(5-4\alpha)}$, the growth of u_Q is under control. Now let us see what happens if $u_Q > 2^{-\frac{j}{2}(5-4\alpha)}$. We rewrite equation (4.1) in terms of wavelet coefficients:

$$\frac{du_Q}{dt} = \sum_{Q', Q'' \in \mathcal{E}(Q)} c_{(Q,Q',Q'')} 2^{\frac{5j(Q)}{2}} u_{Q'} u_{Q''} - 2^{2\alpha j(Q)} u_Q, \tag{4.3}$$

where $\mathcal{E}(Q) := \{PQ, Q\} \cup \mathcal{C}^1(Q)$, and

$$c_{(Q,Q',Q'')} = \left\{ \begin{array}{ll} 1, & \text{if } PQ = Q'' = \tilde{Q} \\ -2^{\frac{5}{2}}, & Q' = Q \text{ and } Q'' \in \mathcal{C}^1(Q) \\ 0, & \text{otherwise} \end{array} \right\}.$$

Having assumed $|u_Q| \gtrsim 2^{-\frac{j}{2}(5-4\alpha)}$ for some time t and assuming that at the initial time $t=0$ it is much smaller, by the smoothness assumption on the initial condition, we integrate (4.3) in time on the interval $[0,T]$ and obtain for one of the choices of (Q', Q'') giving a non-vanishing coefficient:

$$2^{\frac{5j}{2}} \int_0^T |u_{Q'} u_{Q''}| dt \gtrsim 2^{-\frac{j}{2}(5-4\alpha)},$$

which by Cauchy-Schwartz implies:

$$2^{\frac{5j}{2}}(\int_0^T u_{Q'}^2 dt)^{\frac{1}{2}}(\int_0^T u_{Q''}^2 dt)^{\frac{1}{2}} \gtrsim 2^{-\frac{j}{2}(5-4\alpha)},$$

i.e.

$$2^{2j\alpha}(\int_0^T u_{Q'}^2 dt)^{\frac{1}{2}}(\int_0^T u_{Q''}^2 dt)^{\frac{1}{2}} \gtrsim 2^{-j(5-4\alpha)}.$$

This is possible if either

$$2^{2j\alpha}\int_0^T u_{Q'}^2 dt \gtrsim 2^{-j(5-4\alpha)}, \tag{4.4}$$

or

$$2^{2j\alpha}\int_0^T u_{Q''}^2 dt \gtrsim 2^{-j(5-4\alpha)}. \tag{4.5}$$

On the other hand, having in mind conservation of energy we have

$$2^{2j\alpha}\int_0^T \sum_{Q\ at\ scale\ j} u_Q^2 dt \lesssim 1.$$

Thus we conclude that (4.4) or (4.5) could happen in at most $2^{j(5-4\alpha)}$ cubes Q. Now we invoke Lemma 4.1 and conclude that the Hausdorff dimension of the set of points of point at which level of regularity given by

$$u_Q < 2^{-\frac{j}{2}(5-4\alpha)} \tag{4.6}$$

fails is at most $5-4\alpha$.

One still needs to prove regularity on the interior of a dyadic cube Q, provided that one has a little better than critical regularity (4.6) at a cube Q, however we skip details here. The above heuristic was generalized to actual Navier-Stokes equations with hyper-dissipation via localization tools such as Littlewood-Paley operators and pseudo-differential calculus. Although certain technical difficulties needed to be resolved in the transition process from the result addressing model equations to the result for actual equations, the model itself was useful as a device in detecting dimension of the singular set.

We remark that versions of the Navier-Stokes equations with nonlinear modifications in the dissipation term, were proposed and studied too. In 1960s, a few years after Smagorinsky's initial work [43], [44], [45], a general nonlinear viscosity model was introduced by Ladyzhenskaya [20], [21], [22]. The physical motivation for this model came from Kolmogorov's theory of turbulence. The Ladyzhenskaya modified Navier-Stokes equations have the following form:

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\nabla p + div(T(D)) + f, \tag{4.7}$$

$$\nabla\cdot v = 0, \tag{4.8}$$

for $x\in\Omega\subset\mathbb{R}^n$, where the stress tensor T is a function of the n-dimensional version of the velocity gradient

$$D = D(v) = \frac{1}{2}[(\nabla v) + (\nabla v)^T].$$

It is assumed that T satisfies the following properties:

$$(i)\ |T_{ik}(D)| \le c_1(1+|D|^{2\mu})|D|, \tag{4.9}$$

$$(ii)\ T_{ik}(D)(\frac{\partial v_i}{\partial x_k}) \ge \nu_0 D^2 + \nu_1 D^{2+2\mu}, \tag{4.10}$$

(iii) for arbitrary smooth divergence free vectors v' and v'' which are equal on the boundary $\delta\Omega$ the following inequality holds:

$$\int_\Omega (T_{ik}(D') - (T_{ik}(D''))(\frac{\partial v_i'}{\partial x_k} - \frac{\partial v_i''}{\partial x_k})dx \ge \nu_2\int_\Omega \sum_{i,k=1}^{3}(\frac{\partial v_i'}{\partial x_k} - \frac{\partial v_i''}{\partial x_k})^2 dx, \tag{4.11}$$

where c_1, ν_0, ν_1, ν_2 are constants and T_{ik} denotes in entries of the tensor T which are assumed to be continuous functions of $\frac{\partial v_i}{\partial x_k}$.

For $\mu \ge 1/4$, Ladyzhenskaya [22] proved global unique solvability of the boundary value problem for (4.7) - (4.8) in 3 dimensions with T satisfying (i), (ii), and (iii). Following the seminal work of Ladyzhenskaya, further properties of (4.7) - (4.8) with a more general T have been established, see, for example, [31], [41]. In the context of the dyadic models discussed in this article, partial regularity result for a dyadic model reflecting a nonlinear modification in the dissipation term corresponding to $\mu < 1/2$ was studied in [11]. It would be desirable to extend the heuristic dyadic result to the result concerning (4.7) - (4.8).

5. Conclusion

We conclude this note by observing that the dyadic models can, in some instances, be used to detect results which are valid for the Euler and the Navier-Stokes equations, if the results are based on conservation (decay) of energy and scaling. However due to its construction, the dyadic model discussed throughout this paper has the positivity property (which comes as a consequence of (3.1)) that, in turn, implies behavior of the model itself, such as blow-up phenomena, which has not to date been detected in the actual equations for the motion of an incompressible fluid.

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