What is a BPS state?

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1 BPS states

We begin with an account of what a "BPS state" is and what it means to "count" them.

1.1 Hilbert space

The quantum theories we want to understand are describing phenomena which take place in Minkowski space $\mathbb{E}^{3,1}$. Among other things, such a theory is supposed to have an associated Hilbert space $\mathcal{H}$, whose vectors represent the possible "states" of the system.

One simple example of a quantum theory which will be important for us is *quantum Maxwell theory*, which describes the dynamics of the electromagnetic field, perhaps coupled to some dynamical charged particles. In this theory any state can be labeled by a $\mathbb{Z}$-valued *electric* charge (e.g. this charge would be $+3$ in a state with four electrons and one antielectron), and a $\mathbb{Z}$-valued *magnetic* charge (which likewise counts the net number of monopoles). More generally, in the theories of interest for us the Hilbert space is graded by a "charge lattice" Γ:

$$\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma. \tag{1.1}$$

This lattice is equipped with a natural $\mathbb{Z}$-valued antisymmetric pairing $\langle , \rangle$. One convenient way of defining this pairing is as follows: given two particles with charges γ, γ' their crossed electromagnetic fields carry $\langle \gamma, \gamma' \rangle$ units of angular momentum (along the axis from x to x'). Roughly this means that the pairing is nonzero only if γ carries an electric charge and γ' a magnetic charge or vice versa.

1.2 Classification of particles

Our aim is to define invariants $\Omega(\gamma)$ which in an appropriate sense "count 1-particle states with charge γ".

In classical mechanics, the mental picture one usually has of a particle is a little billiard ball that is located at some position and moving with some momentum. In quantum mechanics, it turns out that the sharpest definition of a particle is as a certain *representation*.

Each $\mathcal{H}_\gamma$ is supposed to be a projective unitary representation of the group $G = Isom(\mathbb{E}^{3,1}) = ISO(3,1)$. What does this representation look like?

- The simplest state to consider is the vacuum of the theory. You should think of this state as representing "empty space" (or as close to empty as one can get in the quantum world). It generates a copy of the trivial representation of $ISO(3,1)$ inside $\mathcal{H}$.

- The next simplest kind of state is one where space is empty but for a single particle, propagating with some definite 4-momentum $p \in T^*\mathbb{E}^{3,1} = \mathbb{R}^{3,1}$. The space $\mathcal{H}^1$ of such states decomposes into a discrete sum of representations of $ISO(3,1)$ as follows:

  - First, $\mathcal{H}^1$ splits into components $\mathcal{H}^1_M$ labeled by $M \in \mathbb{R}_{\geq 0}$. M^2 is the eigenvalue of a quadratic Casimir operator in $ISO(3,1)$. Physically it is the mass of the particle.
  - Second, these components have a further discrete decomposition. When $M > 0$ (the only case we will need here), this decomposition arises as follows. We fix some $p_{\mathrm{rest}} \in (\mathbb{R}^{3,1})^*$ with $\|p_{\mathrm{rest}}\|^2 = M^2$, and consider the subspace $\mathcal{H}^{1,\mathrm{rest}}_M \subset \mathcal{H}^1_M$ of states on which the translation subgroup $T = \mathbb{R}^{3,1}$ acts by the character p_{rest}. This space forms a projective unitary representation of the stabilizer of p_{rest},

    $$G_{\mathrm{rest}} = SO(3) \ltimes T \subset ISO(3,1). \tag{1.2}$$

    Now $SO(3)$ has projective unitary representations V_j classified by $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, with $\dim V_j = 2j+1$. So we get a decomposition by type, $\mathcal{H}^{1,\mathrm{rest}}_M = \oplus_j \mathcal{H}^{1,\mathrm{rest}}_{M,j}$. Each $\mathcal{H}^{1,\mathrm{rest}}_{M,j}$ generates a subspace $\mathcal{H}^1_{M,j}$ under the action of the full $ISO(3,1)$. The decomposition of $\mathcal{H}^1_M$ is then

    $$\mathcal{H}^1_M = \oplus_j \mathcal{H}^1_{M,j}. \tag{1.3}$$

  So as far as the symmetries of spacetime are concerned, massive particles are labeled by the two invariants $M \in \mathbb{R}_{\geq 0}$ and $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

- Finally there are the rest of the states, sometimes called the "multiparticle" spectrum. As far as the G-action goes, these states are quite different from the 1-particle states. To understand why, consider the simple case of a two-particle state, composed of two particles with masses M_1 and M_2. By acting with $SO(3,1)$ we can pass to the "center of mass frame" where the momenta are $p_1 = (\sqrt{M_1^2 + P^2}, P, 0, 0)$ and $p_2 = (\sqrt{M_2^2 + P^2}, -P, 0, 0)$, so the total mass is

  $$M = \|p_1 + p_2\|^2 = \sqrt{M_1^2 + M_2^2 + 2\|P\|^2}. \tag{1.4}$$

  Since the relative spatial momentum P can be arbitrary, there is a continuum of possible masses M for the 2-particle state: any $M \in [M_1 + M_2, \infty)$ is allowed. So the 2-particle states do not sit in a single irreducible representation of G or even a discrete sum of them, but rather occur in a continuous family. Similar comments apply to states with more than 2 particles.

1.3 BPS particles

The Hilbert space $\mathcal{H}$ contains a huge amount of information about the quantum field theory, in some sense too much — we would like to distill it down to some more understandable and more invariant quantities. This is hopeless for a general quantum theory, but the situation is better for supersymmetric theories, i.e. quantum theories for which the group $G = ISO(3,1)$ is extended to a supergroup $\tilde{G}$. The particular $\tilde{G}$ that concerns us here has Lie algebra

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_+ \oplus \tilde{\mathfrak{g}}_- \tag{1.5}$$

where the even part $\tilde{\mathfrak{g}}_+$ is a direct sum

$$\tilde{\mathfrak{g}}_+ = \mathfrak{iso}(3,1) \oplus \mathbb{C}. \tag{1.6}$$

The abelian $\mathbb{C}$ factor is central in $\tilde{\mathfrak{g}}$, and has a canonical generator Z.

We can then ask for the decomposition of $\mathcal{H}^1$ under the whole $\tilde{G}$ instead of just under $ISO(3,1)$. The basic approach is closely analogous to the previous section:

- As before, we have the quadratic Casimir M constructed from $ISO(3,1) \subset \tilde{G}$. In addition we now have the central generator Z. These two give a decomposition of $\mathcal{H}^1$ into a discrete sum of pieces $\mathcal{H}^1_{M,Z}$, with $M \in \mathbb{R}_{\geq 0}$ and $Z \in \mathbb{C}$.
- To understand the decomposition of each $\mathcal{H}^1_{M,Z}$, we consider again a subspace $\mathcal{H}^{1,\mathrm{rest}}_{M,Z}$ on which the translations along $\mathbb{R}^{3,1}$ act by some fixed character p_{rest}. It is a representation of a subgroup $\tilde{G}_{\mathrm{rest}} \subset \tilde{G}$,
$$\tilde{G}_{\mathrm{rest}} = SO(3) \ltimes \tilde{T}, \tag{1.7}$$
where the "super translation group" $\tilde{T}$ is generated by the ordinary translations T, plus the central generator Z, plus the "odd translations" $\tilde{\mathfrak{g}}_-$.

  This is where the interesting new wrinkle emerges. In the previous case we only had to consider the ordinary translations, which just act by the fixed character p_{rest} and hence don't contribute much to the representation theory of $\tilde{G}_{\mathrm{rest}}$. In the present case we also have to consider the odd translations, which (once we have diagonalized $\tilde{T}$ and Z) act by a Clifford algebra (on an 8-dimensional vector space). One finds:

  * If $M < |Z|$ there are *no* unitary representations of the Clifford algebra.
  * If $M = |Z|$ the Clifford algebra is degenerate, and its unique unitary irrep S has dimension $2^{4/2} = 4$.
  * If $M > |Z|$ the Clifford algebra is nondegenerate, and its unique unitary irrep S has dimension $2^{8/2} = 16$.

  In either case, since $SO(3)$ acts by automorphisms of the Clifford algebra it automatically acts projectively on S as well, so S is a representation of the whole $\tilde{G}_{\mathrm{rest}}$. As a representation of $SO(3)$, S decomposes as
$$S = \begin{cases} S_{\mathrm{short}} = 2V_0 \oplus V_{\frac{1}{2}} & \text{for } M = |Z|, \\ S_{\mathrm{long}} = 5V_0 \oplus 4V_{\frac{1}{2}} \oplus V_1 & \text{for } M > |Z|. \end{cases} \tag{1.8}$$

The most general irreducible representation of $\tilde{G}_{\text{rest}}$ is obtained by tensoring S with an arbitrary irrep of $SO(3)$, on which $\tilde{T}$ acts trivially. So as before, the irreps of $\tilde{G}_{\text{rest}}$ are labeled by $j \in \frac{1}{2}\mathbb{Z}_+$, but their structure is slightly more complicated: as representations of $SO(3)$ they are

$$\tilde{V}_{j,\text{short/long}} = V_j \otimes S_{\text{short/long}}. \tag{1.9}$$

As before, the decomposition of $\mathcal{H}^1_{M,Z}$ under the full $\tilde{G}$ corresponds to the decomposition of $\mathcal{H}^{1,\text{rest}}_{M,Z}$ under $\tilde{G}_{\text{rest}}$.

1.4 BPS degeneracies

Define $n_j(\gamma) \in \mathbb{Z}_+$ to be the multiplicities occurring in the decomposition

$$\mathcal{H}^1_{\text{rest},\gamma} = \left(\bigoplus_{j \in \frac{1}{2}\mathbb{Z}_+} \tilde{V}_{j,\text{short}}^{\oplus n_j(\gamma)} \right) \oplus (\text{long representations}). \tag{1.10}$$

Naively one might imagine that $n_j(\gamma)$ are the invariants we are after.

However, at least *a priori*, they do not have the desired deformation invariance. The trouble is that as we deform the theory, several short representations can combine into a long one and move off from $M = |Z|$ to $M = |Z| + \epsilon$. To deal with this we define an *index* which counts short representations with some signs (and weights), with the key property that it vanishes for combinations which can combine into long representations. Up to scalar multiple there is a unique such index:

$$\Omega(\gamma) = \sum_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}} (-1)^{2j}(2j+1)n_j(\gamma). \tag{1.11}$$

It is defined more conceptually by

$$\Omega(\gamma) = -\frac{1}{2}\operatorname{Tr}_{\mathcal{H}^1_{\text{rest},\gamma}}(2J)^2(-1)^{2J} \tag{1.12}$$

where $J \in \mathfrak{so}(3)$ is the generator of rotations around any axis.

2 Examples

Perhaps now it is a good time to explain where these Hilbert spaces actually come from in some examples. At the same time, this will explain why any of this business should be of interest for geometers.

2.1 Example: Calabi-Yau threefolds

The starting point is Type II superstring theory. In this theory spacetime is a 10-manifold Y with metric, usually taken to be Lorentzian. We take this 10-manifold to be of the form

$$Y = X \times \mathbb{E}^{3,1} \tag{2.1}$$

where X is a Riemannian six-manifold. Because of the factor $\mathbb{E}^{3,1}$ here this is a quantum system of the type we have been discussing, with a Hilbert space acted on by $ISO(3,1)$. Moreover, if we choose X to be a Calabi-Yau threefold, then the system is supersymmetric.

Now let us consider the space of (charged) 1-particle states. The classical picture of these states is simple to understand: they come from extended objects known as "Dp-branes." One considers such a Dp-brane supported on $D \times c$, where D is a p-cycle in X and c a 1-dimensional timelike curve in $\mathbb{E}^{3,1}$. As far as the spacetime $\mathbb{E}^{3,1}$ is concerned this object looks like a single particle.

The precise story depends on whether we consider Type IIA or Type IIB string theory: in IIA there are branes with any even p, in Type IIB with any odd p.

Now, does this particle give rise to a BPS state or not? To answer this question we consider again the inequality $M \geq |Z|$. The mass M of the particle is just given by the volume of D. The central charge Z turns out to be the integral of an appropriate calibration form. For example, in Type IIB it involves the holomorphic 3-form Ω,

$$Z = \int_D \Omega. \tag{2.2}$$

On the other hand, in Type IIA it is given (in a somewhat oversimplified picture, in particular neglecting quantum corrections) in terms of the Kähler form ω,

$$Z = \int_D \frac{\omega^{p/2}}{(p/2)!} \tag{2.3}$$

In both cases the inequality $M \geq |Z|$ indeed holds, as we expected on general representation-theoretic grounds. Moreover, the Dp-branes which obey $M = |Z|$ are just the ones supported on *calibrated* cycles D. So in Type IIA string theory one can obtain BPS states from holomorphic cycles in X, and in IIB string theory one can obtain BPS states from special Lagrangian 3-cycles.

2.2 Example: M5-brane theories

Here is a second example which is somehow simpler. The starting point is M-theory instead of string theory.

In M-theory spacetime is an 11-manifold, again usually Lorentzian. We take it to be of the form

$$Y = Q \times \mathbb{E}^3 \times \mathbb{E}^{3,1} \tag{2.4}$$

where Q is any hyperkähler 4-manifold.

Then let Σ be a complex submanifold of Q (in one of its complex structures). We introduce an M5-brane supported on $\Sigma \times \mathbb{E}^{3,1}$. (Unlike IIA/B string theory which we considered above, in M-theory the basic objects are relatively few: only M2-brane and M5-brane.) This gives a supersymmetric quantum system of the sort we are after.

In this case the classical picture of the particles we are after is an M2-brane supported on $D \times \{pt.\} \times c$, with c a 1-dimensional timelike curve in $\mathbb{E}^{3,1}$ as before, and D an *open* surface in Q with boundary $\partial D \subset \Sigma$. Their mass is the area of D and their central charge is

$$Z = \int_D \Omega, \tag{2.5}$$

where Ω is the holomorphic symplectic form on Q (much like the Type IIB example above). So the BPS states in this case come from open special Lagrangian surfaces.[1]

2.3 Example: Nonabelian gauge theories

There is a third class of examples of $\mathcal{N} = 2$ supersymmetric theories in $d = 4$. To pick out a member of this class one needs the datum of a compact Lie group G, together with a "sufficiently small" representation R of G obeying some mild conditions.

This class of examples includes a theory famously related to Donaldson's 4-manifold invariants, obtained when we take $G = SU(2)$ and let R be the trivial representation.

3 Walls

In many of these examples there are well-known phenomena in which one of the objects we want to count splits into two as the geometric moduli are varied. This would seem to — and indeed does — imply that the BPS degeneracy $\Omega(\gamma)$ jumps when we vary parameters. So these "invariants" are not as invariant as one might have hoped!

This might seem like a contradiction, since we emphasized that $\Omega(\gamma)$ is invariant under continuous deformations of the $\tilde{G}$-representation $\mathcal{H}^1$. The trouble is with our definition of $\mathcal{H}^1$ itself. We have loosely described it as the space of states which occur *discretely* in $\mathcal{H}$, as opposed to the multiparticle states which occur in direct integrals. One might wonder whether it is really possible to disentangle the two. A closer examination of this question will enlighten us about why wall-crossing phenomena occur.

The BPS states in which we are interested are usually well separated from the direct integrals. To see how this works, consider the space of all states with charge γ. Recall that all such states obey the BPS bound $M \geq |Z_\gamma|$, and the BPS states are the 1-particle states which saturate this bound,

$$M_{\mathrm{BPS}} = |Z_\gamma|. \tag{3.1}$$

Now how about the 2-particle states? Such a state would be made of particles of charges γ_1 and γ_2, with $\gamma_1 + \gamma_2 = \gamma$. From (1.4) recall that these states all have

$$M_{2-\mathrm{particle}} \geq M_1 + M_2. \tag{3.2}$$

Combining this with the BPS bound and the triangle inequality, we find that

$$M_{2-\mathrm{particle}} \geq |Z_{\gamma_1}| + |Z_{\gamma_2}| \geq |Z_{\gamma_1} + Z_{\gamma_2}| = |Z_\gamma| = M_{\mathrm{BPS}}. \tag{3.3}$$

Using (3.3) we see that the continuum of 2-particle states is safely separated from the BPS 1-particle states, *except* in the event that $|Z_{\gamma_1}| + |Z_{\gamma_2}| = |Z_{\gamma_1} + Z_{\gamma_2}|$.

[1]Some readers might prefer to switch to a different complex structure on Q, to get the more conventional setup in which C is special Lagrangian and D is holomorphic. One subtlety here is that there is actually a circle's worth of complex structures for which C is special Lagrangian, but any particular D of interest is only holomorphic in one of them. To get a holomorphic picture of all the D simultaneously you have to consider the *twistor family* of Q, which is of complex dimension 3 and roughly glues together all the complex structures of Q: inside there we could identify the BPS states as holomorphic curves with boundary.

The upshot of all this is that $\Omega(\gamma)$ is well defined *except* when there are two charges γ_1, γ_2 such that $Z_{\gamma_1}/Z_{\gamma_2} \in \mathbb{R}_{\geq 0}$ and $\gamma_1 + \gamma_2 = \gamma$. Since there are only countably many charges γ, if everything were totally generic we would expect that no two of the Z_γ have the same phase. But recall that we are interested in a *family* of quantum systems depending on some parameters. Let $\mathcal{B}$ denote the deformation space parameterizing this family. Then each Z_γ becomes a function $Z_\gamma(u)$ of $u \in \mathcal{B}$, and similarly $\Omega(\gamma)$ is promoted to $\Omega(\gamma; u)$. On $\mathcal{B}$ there are codimension-1 loci ("walls") where some $Z_\gamma(u)$ line up, and in consequence some of the $\Omega(\gamma; u)$ become undefined. Each $\Omega(\gamma; u)$ is *locally* independent of u away from these walls.

4 Wall-crossing formulas

Now a natural question arises. Suppose u_+ and u_- are two points of $\mathcal{B}$ separated by a wall. How are the $\Omega(\gamma; u_+)$ related to the $\Omega(\gamma; u_-)$?

It has recently become apparent that the answer to this question is provided by a remarkable *wall-crossing formula* (WCF), written down by Kontsevich-Soibelman in [1]. To state the formula we need to introduce some strange-looking ingredients (which will be partly demystified below). We begin with the algebraic torus

$$\tilde{T} := \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^\times \tag{4.1}$$

This torus has characters

$$\tilde{X}_\gamma : \tilde{T} \to \mathbb{C}^\times \tag{4.2}$$

obeying

$$\tilde{X}_\gamma \tilde{X}_{\gamma'} = \tilde{X}_{\gamma+\gamma'}. \tag{4.3}$$

[To be continued.]

References

[1] M. Kontsevich and Y. Soibelman, "Stability structures, motivic Donaldson-Thomas invariants and cluster transformations," 0811.2435.