Lectures on BPS states and spectral networks

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Abstract. These are notes for a lecture series on BPS states and spectral networks, for Park City Mathematics Institute, July 2019. They are a bit incomplete so far, missing some references and also missing the last part of the fourth lecture and the fifth lecture. The latest version can be found at http://ma.utexas.edu/users/neitzke/expos/bps-lectures.pdf

Lecture 1: What is a BPS state?

“BPS states” appear very frequently in mathematical applications of quantum field theory. The aim of this lecture is to explain rather generally what a BPS state is and some of their standard properties. In one sentence: we’ll study a representation $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ of a super Lie algebra $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$, and the BPS states are the ones in irreps annihilated by nontrivial subspaces of $\mathcal{A}^1$.

1.1. Quantum mechanics  A time-independent quantum system involves the following data:

- A Hilbert space $\mathcal{H}$,
- A (generally unbounded) Hermitian operator $\mathcal{H} : \mathcal{H} \to \mathcal{H}$.

We think of $\mathcal{H}$ as a generator of the abelian Lie algebra $\text{Lie}(\text{ISO}(0,1)) = \text{Lie}(\text{Isom}(\mathbb{R}^{0,1})) \cong \mathbb{R}$.

Eigenvectors of $\mathcal{H}$ are called bound states, and the corresponding eigenvalues are called bound state energies. Each bound state thus spans a 1-dimensional irreducible representation of $\text{ISO}(0,1)$.

Example 1.1.1 (Particle on the line). In your first course on quantum mechanics you study the “particle on the line.” For this example you need to fix a function $V : \mathbb{R} \to \mathbb{R}$. Then there is a time-independent quantum system $T_1[V]$ with:

- $\mathcal{H} = L^2(\mathbb{R})$,
- $\mathcal{H} = -\frac{1}{2} \frac{d^2}{dx^2} + V$.

Fact 1.1.2. If $V(x)$ is smooth and $V(x) \to \infty$ as $|x| \to \infty$ then $\mathcal{H}$ has discrete spectrum with no accumulation points, and $\mathcal{H}$ has a basis consisting of eigenvectors
of $H$. There is a unique eigenvector with smallest eigenvalue, called the \textit{ground state}.

\textbf{Example 1.1.3 (Harmonic oscillator).} If we take $V = \frac{1}{2}x^2$ in the above then the ground state turns out to be $\psi(x) = e^{-\frac{1}{2}x^2}$, which has $H\psi(x) = \frac{1}{2}\psi(x)$, so that the ground state energy is $\frac{1}{2}$. The bound state energies are $n + \frac{1}{2}, n \in \mathbb{N}$.

\textbf{Remark 1.1.4.} If $V(x)$ is more general, then $H$ need not have a basis of eigenvectors of $H$; e.g. if $V(x) = 0$ there are no $L^2$ eigenvectors at all, although for some purposes the functions $\psi_p(x) = e^{ipx}$ can stand in. These have (formally) $H\psi_p = \frac{p^2}{2}\psi_p$. Thus we say $H$ has \textquotedblleft continuous spectrum\textquotedblright{} consisting of the whole positive real line. More generally we could have some discrete and some continuous spectrum.

We can replace $\mathbb{R}$ by a general Riemannian manifold:

\textbf{Definition 1.1.5 (Laplace operator).} For $M$ a Riemannian manifold, the Laplace operator $\Delta : L^2(M) \to L^2(M)$ is given by

\begin{equation}
\Delta = d^*d
\end{equation}

where $d^*$ is the formal adjoint of $d : L^2(M) \to \Omega^1_{L^2}(M)$.

\textbf{Example 1.1.7 (Particle on a Riemannian manifold).} Fix a Riemannian manifold $M$ and a function $V : \mathbb{R} \to \mathbb{R}$. Then there is a quantum mechanical system $T_1[M]$, the \textquotedblleft particle on $M$\textquotedblright, with:

- $\mathcal{H} = L^2(M)$,
- $H = \frac{1}{2}\Delta + V$.

\textsuperscript{1}$H$ is a self-adjoint operator with compact resolvent.
Now suppose $M$ is compact and we take $V = 0$. In this case, $H = \frac{1}{2} \Delta$ has discrete spectrum bounded below by $0$, and $H \psi = 0$ iff $\psi$ is a constant function. Thus again there is no ground state degeneracy.

1.2. The superparticle

Finally we come to our first supersymmetric example. [8]

Definition 1.2.1 (Laplace operator on differential forms). The Laplace operator $\Delta : \Omega^*_{L^2}(M) \to \Omega^*_{L^2}(M)$ is given by

$$\Delta = [d, d^*] = dd^* + d^*d$$

where $d^*$ is the formal adjoint of $d : \Omega^*_{L^2}(M) \to \Omega^*_{L^2}(M)$.

Example 1.2.3 (Superparticle on a Riemannian manifold). Again fix a Riemannian manifold $M$. Then there is a quantum mechanical system $T_1[M]$, the “superparticle on $M$”, with:

- $H = \Omega^*_{L^2}(M)$,
- $H = \frac{1}{2} \Delta$.

In this case $H$ is $\mathbb{Z}/2\mathbb{Z}$-graded:

$$H = H^0 \oplus H^1, \quad H^0 = \bigoplus_k \Omega^{2k}_{L^2}(M), \quad H^1 = \bigoplus_k \Omega^{2k+1}_{L^2}(M).$$

The key new feature is that in this case there are more operators around than just $H$: $H$ is a unitary $\mathbb{Z}/2\mathbb{Z}$-graded representation of a Lie superalgebra,

Definition 1.2.5 ($N = 2$ supersymmetry in dimension $d = 1$). [2] We define a Lie superalgebra $A$ by

$$A = A^0 \oplus A^1, \quad A^0 = \mathbb{C} \cdot H, \quad A^1 = \mathbb{C} \cdot Q \oplus \mathbb{C} \cdot \overline{Q},$$

ie 2 odd generators $Q, \overline{Q}$ and one even generator $H$, with the brackets \[3\]

$$[Q, Q] = 2H, \quad [Q, Q] = 0, \quad [\overline{Q}, \overline{Q}] = 0, \quad [Q, H] = 0, \quad [H, H] = 0.$$

Indeed $Q$ and $\overline{Q}$ act on $H$ simply by

$$Q = d, \quad \overline{Q} = d^*.$$

\[2\]The $N = 2$ refers to the fact that we have 2 odd generators. The $d = 1$ refers to the fact that $H$ generates $\text{ISO}(0,1)$, the isometries of $\mathbb{R}^{0,1}$ — it does not have to do with the dimension of $M$, which can be anything.

\[3\]Our convention is that $[,]$ means the graded bracket, i.e. for objects $x, y$ which are in grade $n_x, n_y$ respectively, $[x, y] = xy - (-1)^{n_x n_y}yx$. So this bracket is a commutator unless both $x$ and $y$ are odd, in which case it is an anticommutator.
\( \mathcal{Q} \) is adjoint to \( Q \), and \( H \) self-adjoint: we say then that this is a \textit{unitary} representation of \( A \). The unitarity implies that all eigenvalues of \( H \) are nonnegative, because
\[
\langle \psi, H\psi \rangle = \langle \psi, Q\mathcal{Q}\psi \rangle + \langle \psi, \mathcal{Q}Q\psi \rangle = \|Q\psi\|^2 + \|\mathcal{Q}\psi\|^2 \geq 0.
\]
Moreover the norm \( \|\cdot\| \) is nondegenerate, so we conclude that
\[
H\psi = 0 \iff Q\psi = 0, \mathcal{Q}\psi = 0.
\]
In particular, each state with \( H\psi = 0 \) generates a 1-dimensional (trivial) representation of \( A \). We call this representation \( V_0 \) or \( V_1 \) depending whether the single state is in \( \mathcal{H}^0 \) or \( \mathcal{H}^1 \). These representations are called “short.”

On the other hand, if \( H\psi = E\psi \) for some \( E > 0 \), then \( \psi \) generates a 2-dimensional representation of \( A \). These representations are called “long.”

\textbf{Fact 1.2.11.} Each unitary irreducible representation\(^4\) of \( A \) is either long or it is isomorphic to \( V_i \) for some \( i \).

\textbf{Fact 1.2.12.} If \( M \) is compact, then \( \mathcal{H} \) of Equation 1.2.3 is a countable orthogonal direct sum of irreps of \( A \). (We say \( \mathcal{H} \) “contains only discrete spectrum.”)

\[\begin{array}{c}
\mathcal{H}^0 \\
\mathcal{H}^1
\end{array}\]

The short and long irreps have very different character, which we see clearly if we consider \textit{deformations}. For instance, we can imagine continuously varying the Riemannian manifold \( M \). As we do so, the nonzero eigenvalues \( E > 0 \) of \( H \) can change continuously: the long representations are not rigid. On the other hand, the eigenvalues \( E = 0 \) have a harder time changing, because the representations \( V_i \) are rigid! This fact “protects” the ground states. But, it doesn’t protect them \textit{absolutely}: the representation \( V_0 \oplus V_1 \) is \textit{not} rigid, and can deform to a long representation with \( E = \epsilon > 0 \).

\(^4\)By “representation” of \( A \) we always mean a \( \mathbb{Z}/2\mathbb{Z} \)-graded representation.
With that in mind we consider:

**Definition 1.2.13** (Index for representations of $N = 2$ supersymmetry in $d = 1$). The index, or (signed) ground state degeneracy, of a representation $\mathcal{H}$ of $\mathcal{A}$ is

\[(1.2.14) \quad \chi(\mathcal{H}) = (\text{# copies of } V^0_0 \text{ in } \mathcal{H}) - (\text{# copies of } V^1_0 \text{ in } \mathcal{H})\]

The key property of the index is that it is invariant under deformations of $\mathcal{H}$, as long as $\mathcal{H}$ contains only discrete spectrum: if inside $\mathcal{H}$ a copy of $V^0_0 \oplus V^1_0$ deforms into a long representation, then $\chi$ changes by $1 - 1 = 0$.

The main lesson here: while the full Hilbert space $\mathcal{H}$ depends strongly on every little detail of the system, by using a little bit of the representation theory of the supersymmetry algebra $\mathcal{A}$ — looking at representations which are particularly rigid — we are able to extract a more robust invariant. A second lesson: the rigid representations are the ones which are smaller than usual, by virtue of being annihilated by part of $\mathcal{A}$ (in this case actually all of $\mathcal{A}$).

**1.3. Q-cohomology** Another viewpoint on the ground states: in any representation of $\mathcal{A}$ we can define the Q-cohomology,

\[(1.3.1) \quad H^i_Q(W) = \ker Q|_{W^i} / \text{Im } Q|_{W^i}\]

In the case of Equation 1.2.3 the Q-cohomology is something well known: it is the de Rham cohomology of $M$ (with the $\mathbb{Z}$-grading collapsed to a $\mathbb{Z}/2\mathbb{Z}$-grading),

\[(1.3.2) \quad H^i_Q(\mathcal{H}) = H^i_{dR}(M) = \ker d|_{W^i} / \text{Im } d|_{W^i}\]

Now $H^i_Q(\bigoplus_\alpha W_\alpha) \simeq \bigoplus_\alpha H^i_Q(W_\alpha)$ so to compute $H^i_Q(\mathcal{H})$ it’s enough to understand the irreducible components. For $W$ irreducible,

\[(1.3.3) \quad H^i(W) = \begin{cases} W & \text{if } W \simeq V^0_0, \\ 0 & \text{otherwise.} \end{cases}\]

So only the short representations contribute to the Q-cohomology, and each contributes a 1-dimensional space. (This is a $\mathbb{Z}/2\mathbb{Z}$-graded version of the Hodge theorem for compact Riemannian manifolds.)

In particular, the deformation invariant index $\chi$ which we discussed has a simple interpretation:

\[(1.3.4) \quad \chi(\mathcal{H}) = \dim H^0_Q(\mathcal{H}) - \dim H^1_Q(\mathcal{H}) = \dim H^0_{dR}(M) - \dim H^1_{dR}(M)\]
i.e. it is the *Euler characteristic* of $M$!

This view on the cohomology of $M$ was introduced by Witten in the very important paper [8]. There he also considered various deformations of the story, which we won’t have time to treat here. One modification involves adding a Morse function on $M$ (this becomes especially interesting in the limit where the Morse function is very large; one gets a description of the cohomology of $M$ in terms of critical points of the Morse function and gradient flows between them.) Another involves passing to *equivariant* cohomology for a $G$-action on $M$.

**1.4. Richer examples** So far we’ve discussed how cohomology of a compact Riemannian manifold $M$ arises as a space of supersymmetric ground states of a time-independent quantum system $T_1[M]$ — essentially a recapitulation of the usual story of Hodge theory.

We briefly mention a few richer examples of the same structure, to make contact with other lectures at this school. *These are not rigorous statements* but rather accounts of what is claimed in the physics literature.

**Physics Example 1.4.1** (Lagrangian Floer homology). Suppose $M$ is a Kähler manifold. There is a 2-dimensional quantum field theory $T_2[M]$ known as “the supersymmetric sigma model into $M$,” with $N = (2,2)$ supersymmetry. A Lagrangian submanifold $L \subset M$ gives a supersymmetric boundary condition in $T_2[M]$. Now consider $T_2[M]$ in the 2-dimensional spacetime $X = [0,1] \times \mathbb{R}$, with boundary conditions at the two sides induced by $L$ and $L'$.

![Diagram](image)

This system is “effectively 1-dimensional”: its geometric symmetry is just ISO$(0,1)$.

It gives a time-independent quantum system $T_1[M, L, L']$ which is $N = 2$ supersymmetric, so that its Hilbert space $H$ is a representation of $\mathcal{A}$. The space of supersymmetric ground states is conjectured to be isomorphic to (some version of?) *Lagrangian Floer homology* $HF(L, L')$.

**Physics Example 1.4.2** (Khovanov-Rozansky homology). Let $\mathfrak{g}$ be a Lie algebra of ADE type (eg $\mathfrak{g} = \mathfrak{sl}(N)$.) Let $L$ be a link in $\mathbb{R}^3$, and $R$ an irreducible finite-dimensional representation of $\mathfrak{g}$. There is a 6-dimensional quantum field theory $T_6[\mathfrak{g}]$, “$(2,0)$ theory of type $\mathfrak{g}$,” which admits 2-dimensional “defects” of various types, one for each $R$. Theory $T_6[\mathfrak{g}]$ can be formulated on the spacetime $\mathbb{R}^{5,1}$ with a defect inserted along the 2-dimensional submanifold $L \times \{pt\} \times \mathbb{R}^{0,1} \subset \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{0,1}$. 
Here the geometric symmetry remaining is $\text{ISO}(0, 1) \times \text{SO}(2)$. This gives a time-independent quantum mechanical system $T_{1, [g, R, L]}$ which is $N = 2$ supersymmetric,\(^5\) so that its Hilbert space $\mathcal{H}$ is a representation of $A$.

The full $\mathcal{H}$ probably depends on every little detail of $L$, while the ground states should depend only on the isotopy class of $L$. Optimistically, the space of supersymmetric ground states should be well defined, finite-dimensional and isomorphic to the Khovanov-Rozansky homology of $L$. (This is more or less conjectured in [11] Section 5.1.6, in turn a reinterpretation of a proposal in [6].)

In this case there are also two additional $\text{U}(1)$ symmetries in the theory, so the Hilbert space is $\mathbb{Z} \times \mathbb{Z}$-graded, as the KR homology is. One of these symmetries comes from the visible $\text{SO}(2)$, the other comes from an internal “R-symmetry” of the $(2, 0)$ theory.\(^6\)

1.5. Field theory  

Now we consider a more complicated class of examples, which arise from quantum field theory (QFT) in Minkowski space $\mathbb{R}^{d-1,1}$.

Such a QFT is still an example of a time-independent quantum system, and so it determines a Hilbert space $\mathcal{H}$, as before. But if we just study it the way we did before we will see everything is hopelessly infinite: $H$ has no discrete spectrum except maybe the vacuum. Our interest now is not in the vacuum but in “1-particle states,” so we need to organize $\mathcal{H}$ better. We use the fact that instead of $\text{ISO}(0, 1)$ which appeared before, now $\mathcal{H}$ is a unitary representation of a larger group, the Poincare group

\[(1.5.1) \quad \text{ISO}(d - 1, 1) = \text{SO}(d - 1, 1) \times \mathbb{R}^{d-1,1} \subset \text{Isometries}(\mathbb{R}^{d-1,1})\]

or more precisely its universal cover $\text{ISpin}(d - 1, 1)$. So, we need to know a little bit about the unitary representations of $\text{ISpin}(d - 1, 1)$.

The Lie algebra: rotation/boost generators for $\text{so}(d - 1, 1)$ plus translation generators $P^i$ (with $H = P^0$). In the universal enveloping algebra there is a quadratic Casimir operator

\[(1.5.2) \quad p = (P^0)^2 - (P^1)^2 - (P^2)^2 - \ldots - (P^{d-1})^2\]

which thus acts by a scalar in every irreducible representation. It’s convenient to write $M = \sqrt{p}$.

\(^5\)For physicists: a cheap way of getting this counting is to remember from [12] that in $\mathcal{N} = 4$ super Yang-Mills a Wilson loop constrained to a 3-dimensional subspace is $\frac{1}{8}$-BPS, i.e. it preserves 2 of the 16 supercharges.

\(^6\)The $(2, 0)$ theory itself has $\text{SO}(5)$ R-symmetry, but the surface defect on $L \times \mathbb{R}^{0,1}$ breaks that down to $\text{U}(1)$. 

**Fact 1.5.3** (1-particle representations of Poincare group). Given a unitary irreducible representation $S$ of the compact group Spin$(d - 1)$ and a scalar $M > 0$, there is a corresponding unitary irreducible representation $V_{S,M}$ of ISpin$(d – 1,1)$.

Physically $V_{S,M}$ should be thought of as representing the space of states of a massive particle propagating in $\mathbb{R}^{d-1,1}$, with rest mass $M$, and spin governed by the representation $S$. $V_{S,M}$ can be constructed roughly as follows. First we consider a representation of

$$G^\text{rest} = \text{Spin}(d - 1) \rtimes \mathbb{R}^{d-1,1}$$

where Spin$(d – 1)$ acts in the representation $S$ and the translation generators $P^i$ act by the fixed character $(0, \ldots, 0, M)$. Next we extend this representation of $G^\text{rest}$ to the whole ISpin$(d–1,1)$ in the smallest possible way (“induction”) [Mackey...].

Now, a typical structure for the full Hilbert space $\mathcal{H}$ (in theories with a mass gap) is the following:

- There is a unique vacuum state, generating a trivial representation of ISpin$(d – 1,1)$.
- There is a set of “1-particle representations” of the form $V_{S,M_i}$, which occur discretely, with masses $M_1, M_2, M_3, \ldots$, with $M_1 > 0$ (mass gap).
- However, the full $\mathcal{H}$ is not an orthogonal direct sum of irreducible representations; this is true up to any mass $< 2M_1$, but starting at $2M_1$ there is “continuous spectrum” (from multiparticle states).

![Diagram](image.png)

### 1.6. Supersymmetric field theory

Now let’s consider the Hilbert spaces of supersymmetric QFT. These will be representations of a super Lie algebra $\mathcal{A}$, extending Lie(ISpin$(d–1,1)$), much as we did in Equation 1.2.3 above.

The story is different in different dimensions. We’ll focus on two examples.

**Example 1.6.1** ($N = (2,2)$ supersymmetry in $d = 2$). The supersymmetry algebra $\mathcal{A}$ in this case has 4 odd generators,

$$Q^\pm, \quad \overline{Q}^\pm$$

and 7 even generators,

$$P^0, \quad P^1, \quad Z, \quad \overline{Z}, \quad \text{so}(1,1)$$
Z and Z are central ("central charges"). The nonvanishing commutation relations of the odd generators are

\[(1.6.4)\quad [Q^+, \overline{Q}^+] = P^+, \quad [Q^+, Q^-] = Z,\]
\[(1.6.5)\quad [Q^-, \overline{Q}^-] = P^-, \quad [\overline{Q}^+, Q^-] = Z,\]

where we defined
\[(1.6.6)\quad P^\pm = P_0^\pm + P_1^\pm = H_\pm + P_1^\pm.\]

Now we want to study representations of \(A\). As before we start by considering the subalgebra \(A^{\text{rest}}\), generated by \(Q^\pm, \overline{Q}^\pm, P_0, P_1, Z, \overline{Z}\). We fix some \(M > 0\) and \(Z \in \mathbb{C}\), and consider a representation of \(A^{\text{rest}}\) where \(P = (P_0, P_1)\) acts by the character \((M, 0)\) for some \(M > 0\), and \(Z\) acts by the scalar \(Z\). Then our relations become

\[(1.6.7)\quad [Q^+, \overline{Q}^+] = M, \quad [Q^+, Q^-] = Z,\]
\[(1.6.8)\quad [Q^-, \overline{Q}^-] = M, \quad [\overline{Q}^+, Q^-] = Z,\]

Now consider the generator
\[(1.6.9)\quad Q_{\vartheta} = \frac{1}{\sqrt{2}} \left( e^{i\vartheta/2} Q^+ + e^{-i\vartheta/2} \overline{Q}^- \right).\]

This has
\[(1.6.10)\quad [Q_{\vartheta}, \overline{Q}_{\vartheta}] = M + \frac{1}{2} (e^{i\vartheta Z} + e^{-i\vartheta \overline{Z}}).\]

We optimize by choosing \(\vartheta = \pi - \arg Z\). Then
\[(1.6.11)\quad [Q_{\vartheta}, \overline{Q}_{\vartheta}] = M - |Z|.\]

But now
\[(1.6.12)\quad (M - |Z|) \|\psi\|^2 = \langle \psi, [Q_{\vartheta}, \overline{Q}_{\vartheta}] \psi \rangle = \|Q_{\vartheta} \psi\|^2 + \|\overline{Q}_{\vartheta} \psi\|^2 \geq 0\]

then implies the bound
\[(1.6.13)\quad M \geq |Z|\]

which moreover is saturated if and only if
\[(1.6.14)\quad Q_{\vartheta} \psi = \overline{Q}_{\vartheta} \psi = 0.\]

The nature of the representations depends on whether the bound (1.6.13) is actually saturated. Indeed, the relations (1.6.7)-(1.6.8) say that the four Q’s make up a Clifford algebra:

- When \(M > |Z|\) this Clifford algebra is nondegenerate, and its irreducible representations have dimension \(2^{4/2} = 4\).\(^7\)
- When \(M = |Z|\) we’ve seen that \(Q_{\vartheta}\) and \(\overline{Q}_{\vartheta}\) act identically as zero, and the Clifford algebra is degenerate: its irreducible representations have only dimension \(2^{2/2} = 2\).

\(^7\)Recall that the nondegenerate complex Clifford algebra of dimension \(2n\) has a unique irreducible representation, which has dimension \(2^{n/2}\).
By induction as before, these representations of $\mathcal{A}^{\text{rest}}$ get extended to representations of the whole algebra $\mathcal{A}$.

So far so good, but in order to define the index we need a small further hypothesis: we extend $\mathcal{A}$ by an even generator $F$ ("fermion number") which extends the $\mathbb{Z}/2\mathbb{Z}$-grading to a $\mathbb{Z}$-grading, obeying $[F, Q^\pm] = Q^\pm$ and $[F, Q^\mp] = Q^\mp$. After so doing, we have: [add a few more details here]

**Fact 1.6.15** (1-particle representations of $N = (2, 2)$ supersymmetry in $d = 2$).

Given a complex scalar $Z \neq 0$ and a real scalar $M \geq |Z|$, there are two corresponding unitary irreducible representations $V^0_{Z, M}$, $V^1_{Z, M}$ of $\mathcal{A}$.

The representations $V^i_{Z, M} = |Z|$ are called short or "BPS", while the $V^i_{Z, M}$ with $M > |Z|$ are long. The short representations are separately rigid, but $V^0_{Z, M} \oplus V^1_{Z, M}$ can deform to either $V^i_{Z, M}$. Thus we define a new index:

**Definition 1.6.16** (Index for representations of $N = (2, 2)$ supersymmetry in $d = 2$).

$$\mu(\mathcal{H}) = (# \text{ copies of } V^0_{Z, M = |Z|} \text{ in } \mathcal{H}) - (# \text{ copies of } V^1_{Z, M = |Z|} \text{ in } \mathcal{H})$$

(1.6.17) $$= \text{Tr}_{\mathcal{H}^{\text{BPS}}} (-1)^F. \tag{1.6.18}$$

**Remark 1.6.19** (The phase of the central charge). Note that the phase of $Z$ plays a critical role in this story: it determines which 2-dimensional subspace of the 4-dimensional $\mathcal{A}_1$ annihilates the BPS states.

**Remark 1.6.20** (Charge gradings). We will typically consider situations in which the Hilbert space $\mathcal{H}$ has a further grading by "charges"

(1.6.21) $$\mathcal{H} = \bigoplus_a \mathcal{H}_a$$

with an additive structure, such that in each sector $\mathcal{H}_a$ the central charge acts by a constant $Z_a \in \mathbb{C}$, and $Z_{a+b} = Z_a + Z_b$. In this situation we can define $\mu(\mathcal{H}_a)$ for each sector separately.

**Remark 1.6.22** (Deformation invariance and wall-crossing). Formally it looks like the indices $\mu(\mathcal{H}_a)$ deserve to be invariant under deformations of $\mathcal{H}_a$, as was the index in Equation 1.2.13, but there is a hitch: as we noted before, in QFT $\mathcal{H}$ always contains continuous spectrum, and this could spoil our arguments. As long as the continuum stays bounded away from $M = |Z|$ we will be safe (we can just restrict to the part of the Hilbert space below the continuum), but if it touches $M = |Z|$ we may be in trouble.

As a rough diagnostic for whether this will happen one can consider a 2-particle state: take two particles in a common rest frame, and imagine we can neglect their interaction (say by placing them very far apart). This state will have total $M = M_1 + M_2$ and $Z = Z_1 + Z_2$. Now,

(1.6.23) $$M = M_1 + M_2 \geq |Z_1| + |Z_2| \geq |Z_1 + Z_2| = |Z|$$
with equality iff both particles are BPS and \( \arg Z_1 = \arg Z_2 \). Thus the 2-particle states usually are safely separated from the 1-particle BPS states: the only exception is a state of 2 particles which are themselves BPS and have \( \arg Z_1 = \arg Z_2 \).

This leads to the expectation that \( \mu(H_a) \) will be invariant under deformations, except that it may jump when the central charges \( Z_b \) and \( Z_c \) of two BPS particles become aligned, where \( a = b + c \). This is a kind of “codimension 1” phenomenon in parameter-space, so \( \mu(H) \) may jump at some codimension 1 walls in parameter-space. That is indeed what one finds (and we will see it in the following lectures).

**Example 1.6.24** (\( N = 2 \) supersymmetry in \( d = 4 \)). Now we consider 4-dimensional theories. In this case the supersymmetry algebra \( A \) extending \( \text{Lie}(\text{Spin}(3,1)) \) has 8 odd generators, and again central even generators \( Z, \bar{Z} \). See the exercise sheet for detailed formulas. The analysis of representations runs parallel to the above, with the following outcome:

**Fact 1.6.25** (1-particle representations of \( N = 2 \) supersymmetry in \( d = 4 \)). Given a complex scalar \( Z \neq 0 \), a real scalar \( M \geq |Z| \), and an irreducible representation \( S \) of \( \text{Spin}(3) \simeq \text{SU}(2) \), there is a corresponding unitary irreducible representation \( V_{Z,M,S} \) of \( A \).

Again the representations with \( M = |Z| \) are short or “BPS” — annihilated by 4 of the 8 supercharges — while those with \( M > |Z| \) are long. The short representations are individually rigid, but can combine to make long representations. The only index available is the “second helicity supertrace,”

\[
\Omega(H) = \sum_{n \geq 1} (-1)^{n+1} (\# \text{ copies of } V_{Z,M=|Z|,S_n} \text{ in } H)
\]

where \( S_n \) denotes the (unique up to isomorphism) irreducible \( n \)-dimensional representation of \( \text{Spin}(3) \). For example,

\[
\Omega(V_{Z,|Z|,S_1}) = +1 \quad \text{“massive hypermultiplet”}
\]

\[
\Omega(V_{Z,|Z|,S_2}) = -2 \quad \text{“massive vector multiplet”}
\]
Lectures 2-3: 2d theories, tt* geometry and Stokes phenomenon

Today’s lecture is about \( N = (2, 2) \) supersymmetric theories in \( d = 2 \). Such theories come in families parameterized by complex manifolds.\(^8\) These families carry a rich geometric structure, revealed in a large body of literature: pioneering early work is Witten [9], Dijkgraaf-Verlinde-Verlinde [3], Cecotti-Vafa [1, 2], Dubrovin [4], [add more modern refs]

There are two basic classes of examples: “Landau-Ginzburg models” which we’ll discuss, and “quantum cohomology” which we won’t discuss (sorry). [should add refs]

**Physics Fact 2.0.1.** Suppose given a Kähler manifold \( M \) and a holomorphic function \( W \) on \( M \). There is a 2-dimensional quantum field theory \( T^2[M,W] \) called the Landau-Ginzburg model on \( M \), with \( N = (2, 2) \) supersymmetry.

In this lecture we will just consider the case where \( M = \mathbb{C}^k \) (usually \( k = 1 \)) with its standard metric and \( W \) polynomial. We’ll be interested in what happens as we vary the coefficients of \( W \).

**Example 2.0.2 (Potentials for the cubic LG model).** We take \( C = \mathbb{C} \) and for any \( z \in \mathbb{C} \):
\[
W_z(x) = \frac{1}{3} x^3 - zx.
\]

**Example 2.0.4 (Potentials for the quartic LG model).** We take \( C = \mathbb{C}^2 \) and for any \( z = (z_1, z_2) \in \mathbb{C} \):
\[
W_z(x) = \frac{1}{4} x^4 - \frac{1}{2} z_1 x^2 - z_2 x.
\]

2.1. BPS solitons

**Definition 2.1.1 (Classical BPS solitons in Landau-Ginzburg models).** Suppose given a superpotential \( W(x^1, \ldots, x^k) \) with all critical points nondegenerate. A classical BPS soliton with phase \( \vartheta \) is a map \( x : \mathbb{R} \to \mathbb{C}^k \) obeying
\[
\frac{dx^j}{ds} = e^{i\vartheta} \frac{\partial}{\partial x^j} W(x(s))
\]

Given two critical points \( x_i, x_j \) of \( W \) (vacua), an \( ij \)-soliton is a BPS soliton with \( \lim_{s \to -\infty} x(s) = x_i, \lim_{s \to +\infty} x(s) = x_j \).

\(^8\)Actually the structure can be more general, but we’ll simplify by considering only the “chiral deformations” and not the “twisted chiral” ones.
If (2.1.2) is satisfied we have \( \frac{dW(x(s))}{ds} = i \theta |\nabla W(x(s))|^2 \). Thus composing with the map \( W \) projects a classical BPS soliton to a line segment in \( \mathbb{C} \); an \( ij \)-soliton projects to a segment connecting the critical values \( W(x_i) \) and \( W(x_j) \).

To study the equation (2.1.2) explicitly is hard, but if we don’t keep track of the parameterization it becomes easier: by considering the local model \( W = \sum_{i=1}^{k} x_i^2 \) near a critical point we see that above each point \( W \) along the segment there are two “vanishing cycles” \( V_i \) and \( V_j \), each with the topology of \( S^{k-1} \), and \( ij \)-solitons correspond to points of \( V_i \cap V_j \).

**Definition 2.1.3** (Soliton degeneracies in Landau-Ginzburg models). Given two critical points \( x_i, x_j \) of \( W \), we define \( \mu(i,j) \) to be the (signed) intersection number between \( V_i \) and \( V_j \).

**Physics Fact 2.1.4.** The \( \mu(i,j) \) are BPS indices in the \( N=(2,2) \) supersymmetric Landau-Ginzburg model. (More precisely: the Hilbert space in this theory is divided up into sectors \( \mathcal{H} = \bigoplus_i \mathcal{H}_{ij} \), and \( \mu(i,j) = \mu(\mathcal{H}_{ij}) \) in the sense of (1.6.17) above.)

There is a subtlety: how to orient the vanishing cycles? Without explaining this, what we have given is really only a definition of \( |\mu| \), not \( \mu \). I’m not going to explain this: see Gaiotto-Moore-Witten [5]! Likewise, I’m not going to describe the actual vector spaces: again see Gaiotto-Moore-Witten [5]!

**Example 2.1.5** (Soliton degeneracies in the cubic LG model). In this case the structure is very simple: \( V_i \) and \( V_j \) are distinct 2-element subsets of a 3-element set, so \( |V_i \cap V_j| \) is always 1, i.e. for any \( z \neq 0 \) we have \( \mu(1,2; z) = \pm 1 \) and \( \mu(2,1; z) = \mp 1 \).

**Example 2.1.6** (Soliton degeneracies in the quartic LG model). This case is already more interesting: as we change \( z_1 \) and \( z_2 \) the \( \mu(i,j;z_1,z_2) \) can change. The change occurs when three critical values of \( W \) become collinear. In one region there are two solitons, in another region there are three; e.g. in the slice \( z_1 = -1 \), the picture in the \( z_2 \)-plane looks like:

---

9To be exact, see Section 12.3, page 266. The basic idea: realize solutions of (2.1.2) as critical points of a certain Morse function in infinite dimensions, then promote them to generators of a Morse complex with differentials given by gradient flows. cf. simpler example: critical points of a Morse function on a finite-dimensional manifold can be promoted to cohomology vector spaces by the same construction.
The boundary between the two regions is called a “wall of marginal stability.” This is our first example of the wall-crossing phenomenon we mentioned in the first lecture. What happens concretely to the 13-soliton that disappears at the wall? As we approach the wall from the outside, the soliton starts to look more and more like a combination of a 12-soliton and a 23-soliton separated by a large distance $\Delta s$. This large distance $\Delta s \to \infty$ as $z$ approaches the wall.

### 2.2. Wall-crossing formula and spectral networks for 2d $\mathcal{N} = (2, 2)$ theories

The heuristic picture above suggests that the wall-crossing behavior should be governed by a universal law, as follows. Let

\[(2.2.1) \quad Z_{ij} = W_i - W_j.\]

When $Z_{ij}$ and $Z_{jk}$ become collinear, $\mu$ changes by a jump of the form [2]

\[(2.2.2) \quad \mu(i, k) \to \mu(i, k) \pm \mu(i, j)\mu(j, k)\]

This law turns out to be true in $\mathcal{N} = (2, 2)$ theories much more generally. We are going to sketch a route to proving it, but first let’s explore how it lets us determine $\mu$ in practice.

**Definition 2.2.3** (Spectral network for 2d $\mathcal{N} = (2, 2)$ theory). For any $\vartheta$, the spectral network $\text{SN}(\vartheta) \subset \mathbb{C}$ is the set of all $z \in \mathbb{C}$ such that there exists some critical points $x_i, x_j$ with $\mu(i, j; z) \neq 0$ and $\arg Z_{ij} = \vartheta$.

**Example 2.2.4** (Spectral network for the cubic LG model). In this case the critical values are $W(x_i) = \pm \frac{2}{3}z^2$. There will be a soliton with phase $\vartheta = 0$ just if the two critical values have the same imaginary part, i.e. just when $z^2$ is real. This gives the picture of $\text{SN}(\vartheta = 0)$:
In the second picture we chose a branch cut for the function $z^3$ and labeled the two critical values as $x_1$, $x_2$. The walls are labeled by the $ij$ type of the soliton with phase $\vartheta = 0$. The orientation shows the direction in which the mass $M = |W(x_i) - W(x_j)| = \frac{4}{3}|z^{3/2}$ of the soliton increases.

**Example 2.2.5** (Spectral network for the quartic LG model). In this case the full picture would be drawn on $C = C^2$. Again let’s take the slice $z_1 = -1$. Here is the picture in the $z_2$-plane, at phase $\vartheta = \frac{\pi}{2}$:

![Diagram showing spectral network](image)

(This picture and many other similar ones can be found in [7].)

The appearance of the third trajectory at the intersection point is a manifestation of wall-crossing. (Note that the intersection point lies on the wall of marginal stability; varying $\vartheta$ this point would sweep out the whole wall.)

After restriction to the slice, the trajectories in $SN(\vartheta)$ are solutions of a differential equation:

\[
\frac{dZ_{ij}}{dt} = e^{i\vartheta}.
\]

In fact there’s an algorithm for determining $SN(\vartheta)$ for any $\vartheta$, and thus determining all the $\mu(i,j;z)$, as follows. First look at the branch locus of $\Sigma \to C$; this is where some $Z_{ij} \to 0$. Near this locus, we will have $\mu(i,j;z) = \pm 1$ as in Equation 2.2.4 above. Thus the local structure of $SN(\vartheta)$ around each branch point is 3-pronged as in that example. So first we draw these 3 walls coming out of each branch point, and then we use (2.2.6) to continue the walls from there. When walls cross
one another they can give birth to additional walls as shown above: an $ij$ wall and a $jk$ wall can generate a new $ik$ wall.

### 2.3. Chiral rings and vacua

**Physics Fact 2.3.1** (Chiral ring structure). Suppose $C$ is the parameter space of chiral deformations of an $N=(2,2)$ supersymmetric field theory. Then $C$ carries:

- a holomorphic vector bundle of commutative algebras $E$ over $C$ (chiral rings),
- a holomorphic map of vector bundles $q: TC \to E$.

**Remark 2.3.2** (Vacua). Dually, sometimes it is convenient to consider the spectrum $\Sigma_x$ of the commutative algebra $E_x$ instead of the algebra itself. The points of $\Sigma_x$ are (in 1-1 correspondence with) the vacua of the theory; they give a space $\Sigma \to C$ (in our examples it will be a branched covering).

**Remark 2.3.3** (A chiral ring structure gives a Higgs bundle). If we define $\varphi: TC \to \text{End}(E)$ by $(\varphi(v))(w) = q(v) \cdot w$ then the pair $(E, \varphi)$ define a Higgs bundle over $C$.

**Example 2.3.4** (Chiral rings in Landau-Ginzburg models). Suppose given a family of complex polynomials $W_z(x^1, \ldots, x^k)$ parameterized by a space $C$.

Then the Landau-Ginzburg model gives the following structures:

- $E_z$ is the “Jacobian ring” $E_z = \mathbb{C}[x^1, \ldots, x^k]/(\partial_{x^1}W_z, \ldots, \partial_{x^k}W_z)$.
- The map $q$ takes a vector $\partial_z \in TC$ to $\partial_z W_z \in E$.

$\Sigma_z$ consists of the critical points of $W_z$, i.e., solutions of $\partial_{x^i}W_z = 0$.

**Example 2.3.5** (Chiral rings in the cubic LG model). We consider the LG model with a family of potentials parameterized by $C = \mathbb{C}$:

$W(x) = \frac{1}{3}x^3 - zx$.

Then $E$ has a basis $(1, x)$ and the algebra structure is given by the relation $x^2 = z$, from which we obtain

$\varphi(\partial_z) = (-x \cdot) = -\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$.

The vacua make a double cover $\Sigma \to C$, branched over $z = 0$.

**Example 2.3.8** (Chiral rings in the quartic LG model). We consider the LG model with the potentials parameterized by $C = \mathbb{C}^2$:

$W(x) = \frac{1}{4}x^4 - \frac{1}{2}z_1x^2 - z_2x$.

---

10 In the sense of Schaposnik’s lectures, except that here we allow $C$ to be higher-dimensional; then the definition of Higgs bundle requires the extra condition $[\varphi, \varphi] = 0$, which is satisfied here because the algebra structure on $E$ is commutative.

11 I’m not sure of the precise conditions one should put on this family of $W$. 
Now E has a basis \( \{1, x, x^2\} \) and the relation \( x^3 = z_1 x + z_2 \), so

\[
(2.3.10) \quad \varphi(\partial_{z_1}) = \left( -\frac{1}{2} x^2 \right) = -\frac{1}{2} \begin{pmatrix} 0 & z_2 & 0 \\ 0 & z_1 & z_2 \\ 1 & 0 & z_1 \end{pmatrix}, \quad \varphi(\partial_{z_2}) = (-x \cdot) = - \begin{pmatrix} 0 & 0 & z_2 \\ 1 & 0 & z_1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

The vacua make a triple cover \( \Sigma \to \mathbb{C}^2 \), branched over the curve \( 2z_3^3 = 27z_2^2 \).

2.4. The topological connections

A family of \( \mathcal{N} = (2, 2) \) theories also has more structure (ultimately derived from the possibility of making a family of 2d TFTs by “twisting”): there are

- a bilinear pairing \( \langle \cdot, \cdot \rangle \) on \( E \) such that \( \langle a, bc \rangle = \langle ab, c \rangle \),
- a pencil of flat connections \( \nabla^h \) in \( E \), of the form

\[
(2.4.1) \quad \nabla^h = \nabla^\infty + h^{-1}\varphi,
\]

such that \( \langle \cdot, \cdot \rangle \) is compatible with \( \nabla^\infty \). This is a somewhat nontrivial structure, as you will discover if you try to construct it by hand! This point of view is taken most explicitly in [4]. [Saito, Dubrovin, WDVV, CV, ...]

**Example 2.4.2** (Topological connection in the cubic LG model). In the cubic LG model, in the basis \( \{1, x\} \), the pairing and connections are

\[
(2.4.3) \quad \langle \cdot, \cdot \rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \nabla^h_z = \partial_z - h^{-1} \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}.
\]

The flat sections can be written “explicitly” in terms of Airy functions (indeed this connection would be obtained from the Airy equation \( (h^2 \partial^2_x + z)\psi(z) = 0 \) by the standard device of writing a second-order equation as a matrix first-order equation.)

**Example 2.4.4** (Topological connection in the quartic LG model). In the quartic LG model, in the basis \( \{1, x, x^2\} \), the pairing and connections are

\[
(2.4.5) \quad \langle \cdot, \cdot \rangle = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & z_1 \end{pmatrix},
\]

\[
(2.4.6) \quad \nabla^h_{z_1} = \partial_{z_1} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{h^{-1}}{2} \begin{pmatrix} 0 & z_2 & 0 \\ 0 & z_1 & z_2 \\ 1 & 0 & z_1 \end{pmatrix}, \quad \nabla^h_{z_2} = \partial_{z_2} - h^{-1} \begin{pmatrix} 0 & 0 & z_2 \\ 1 & 0 & z_1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

In this example you see that there is a tricky extra term in \( \langle \cdot, \cdot \rangle \) and \( \nabla^h_{z_1} \), which would have been hard to guess (but without it nothing works!)
Happily, there is a construction that produces this kind of formula automatically. The pairing is
\[ \langle f, g \rangle = \text{Res}_{x=\infty} \frac{f(x)g(x)}{W'(x)}. \]
and there is a (slightly tricky) way of getting \( \nabla h \) as well [refs].

At any rate, once you have somehow got the connection \( \nabla h \), a basic question is: what are the flat sections? We can produce some \( \psi_i(z) \) that do the job, by giving their pairing with a general \( \alpha \in E_z \): represent \( \alpha \) by a polynomial \( \alpha(x) \) of degree \( \leq (\deg W) - 2 \), and then do an explicit contour integral in the x-plane,
\[ \langle \psi_i(z), \alpha \rangle = \int_{C_i} \ dx \ e^{h^{-1}W_z(x)} \alpha(x). \]

The remaining trouble is to find the contours \( C_i \). We need \( C_i \) to have no boundary, but we also need the integral to be convergent. A systematic way to get a contour \( C_i \) is: start from a critical point \( x_i \) of \( W \), then follow the paths of steepest descent for the integrand \( h^{-1}W(x) \), i.e. paths along which \( \text{Im}(h^{-1}W) \) is constant while \( \text{Re}(h^{-1}W) \to -\infty \). These are the simplest examples of “Lefschetz thimbles.”

\[ \text{Remark 2.4.9 (Lefschetz thimbles as boundary conditions).} \] The Lefschetz thimbles \( C_i \) provide natural supersymmetric boundary conditions in the Landau-Ginzburg model. Then the flat sections \( \psi_i \) appear naturally as the “boundary states” created by these boundary conditions.

The contour \( C_i \) is well defined unless it runs into another critical point \( x_j \). The latter can happen only if there is a soliton connecting \( x_i \) to \( x_j \), with phase \( \theta = \arg h \). When such a soliton exists, the contour \( C_i \) will jump.

\[ W = \frac{x^3}{3} - 2x \]

The contour \( C_i \) is well defined unless it runs into another critical point \( x_j \). The latter can happen only if there is a soliton connecting \( x_i \) to \( x_j \), with phase \( \theta = \arg h \). When such a soliton exists, the contour \( C_i \) will jump.

\[ ^{12} \text{There is a similar recipe in more general examples, but the details can be considerably more elaborate once we get beyond } W(x) \text{ polynomial in one variable. [Saito, primitive forms]} \]
\[ ^{13} \text{cf. Pavel Putrov’s lectures which involved a much more sophisticated example: there the superpotential } W \text{ was defined on the infinite-dimensional space of connections on a 3-manifold } M, \text{ and the critical points } x_i \text{ were flat connections. To get a 2-dimensional } N = (2,2) \text{ supersymmetric theory with this } W, \text{ one should presumably start with 5-dimensional supersymmetric Yang-Mills theory and compactify it on } M \ [10]. \]
So we arrive at the following picture: in each domain on the complement of the spectral network $\text{SN}(\theta = \arg h)$ we have a basis of flat sections $\psi_i(z)$, but the basis can change when $z$ crosses $\text{SN}(\theta = \arg h)$. This picture recurs more generally in LG models. It turns out that when we move from one domain to another across a wall with label $ij$, the basis transforms by a matrix

$$S = 1 \pm \mu(i, j; z)e_{ij} \tag{2.4.10}$$

(where $e_{ij}$ is the elementary matrix).

Now we can revisit the wall-crossing formula for the soliton counts $\mu(i, j; z)$. We consider the changes of basis induced by traveling two different paths in $\mathbb{C}$:

Requiring that these two changes of basis are equal leads to the following:

$$\begin{pmatrix} 1 & \mu(1, 2) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \mu(1, 3) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu(2, 3) \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu'(1, 2) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu'(1, 3) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu'(1, 2) & 0 \\ 1 & 0 & \mu'(1, 3) \\ 0 & 1 & 0 \end{pmatrix} \tag{2.4.11}$$

This holds just if

$$\mu'(1, 2) = \mu(1, 2), \quad \mu'(2, 3) = \mu(2, 3), \quad \mu'(1, 3) = \mu(1, 3) + \mu(1, 2)\mu(2, 3) \tag{2.4.12}$$

which is the wall-crossing formula we stated before!

**Remark 2.4.13** (Spectral network as Stokes graph). The construction of the $\psi_i(z)$ by contour integration allows one to determine their complete asymptotic series expansion as $h \to 0$ with $\arg h = \theta$, by the method of steepest descent (eg the leading behavior is like $\exp(W(x_i)/h)$). This asymptotic series only depends on what happens very near the critical point; in particular, it does not change when the contours $C_i$ jump. (Another way of saying this is that when the jump occurs, it shifts the contour by adding the contribution of a contour through a subdominant critical point, which thus doesn’t change the leading asymptotic.)

Thus the local solutions $\psi_i(z)$ have the nice property that they have a simple uniform asymptotic expansion as $h \to 0$ with $\arg h = \theta$. It turns out that this property implies that they have to jump somewhere! This is (a manifestation of) the Stokes phenomenon.
In a sense I won’t make precise here, the particular curves that make up \( SN(\theta) \) are the most canonical choice for the location of the jumps. This location is sometimes called Stokes graph. [refs]

2.5. \( tt^* \) geometry

Another crucial discovery of Cecotti-Vafa [1] is that \( N = (2, 2) \) theories have a controllable metric structure.

**Definition 2.5.1 (\( tt^* \) structure).** A \( tt^* \) structure over \( C \) is a chiral ring structure plus a Hermitian metric \( h \) in \( E \), obeying the \( tt^* \) equation,

\[
F_D h + [\varphi, \varphi^h] = 0.
\]

(2.5.2)

Here \( D_h \) is the Chern connection uniquely determined by compatibility with both the holomorphic structure and the Hermitian metric in \( E \).

**Example 2.5.3 (The \( tt^* \) structure for the cubic LG model).** In the cubic LG model of Equation 2.0.2 there is a \( tt^* \) structure for which the metric \( h \) is diagonal: making the ansatz

\[
h = \begin{pmatrix} e^{-u} & 0 \\ 0 & e^u \end{pmatrix}
\]

(2.5.4)

the \( tt^* \) equation becomes

\[
\partial_z \partial_{\bar{z}} u - (e^{2u} - e^{-2u}|z|^2) = 0
\]

(2.5.5)

which has a unique smooth solution \( u(|z|) \), expressible in terms of Painleve transcendents. This turns out to be a very useful “model solution” for studying the behavior of solutions of Hitchin’s equations when one goes to infinity in the moduli space [GMN, MSWW, Fredrickson, ...]

**Definition 2.5.6 (The improved connections).** We define a family of connections \( \nabla^\zeta \) in the bundle \( E \) over \( C \) by

\[
\nabla^\zeta = D_h + \zeta^{-1} \varphi + \zeta \varphi^h
\]

(2.5.7)

The \( tt^* \) equation implies that \( \nabla^\zeta \) is flat for every \( \zeta \in \mathbb{C}^\times \).

The picture we described above for the topological connections \( \nabla^h \) holds equally for the connections \( \nabla^\zeta \): there are flat sections in each domain, given by a formula like (2.4.8),

\[
\langle \psi_i(z), \alpha \rangle = \int_{C_i} dx \, e^{\zeta^{-1} W_\zeta(x) + \zeta \overline{W_\zeta(x)}} \alpha(x),
\]

(2.5.8)

but in this case the representatives \( \alpha(x) \) we choose don’t have a simple explicit expression. Nevertheless the formal structure is the same: in particular these sections also transform by (2.4.10) at walls.

[explain how to use \( \mu \) and \( W \) to compute these sections, a la Dubrovin / CV]

---

\[14\]Note that this is also the Hitchin equation appearing in Schaposnik’s lectures!
Lecture 4: 4d theories and spectral networks

We have discussed some rich structure which appears on \( C \) when \( C \) is a moduli space of \( N = (2,2) \) supersymmetric field theories: in particular \( C \) carries

- a Higgs bundle \((E, \varphi)\),
- two distinguished families of flat connections \( \nabla^h \) and \( \nabla^i \) in \( E \),
- BPS state indices and a spectral network, which control the Stokes phenomena of these connections.

The kind of \( C \) we met so far were very simple: just vector spaces. But it turns out that this kind of structure exists far more generally: for example, we can take \( C \) to be any Riemann surface! That is what we will discuss now.

One of our main aims is to understand the moduli space of Higgs bundles on \( C \); but for this lecture we mainly focus on one Higgs bundle at a time.

3.1. Class \( S \) and surface defects

The physical underpinnings of the construction depend on facts we mentioned in Equation 1.4.2, also mentioned in Pavel Putrov’s last lecture:

**Physics Fact 3.1.1.** There is a 6-dimensional supersymmetric QFT \( T_6[g] \), depending only on the data of an ADE Lie algebra \( g \). This theory supports a 2-dimensional defect labeled by an irreducible finite-dimensional representation \( R \) of \( g \). [refs]

Now suppose given a Riemann surface \( C \). We consider the theory \( T_6[g] \) on the spacetime \( C \times \mathbb{R}^{3,1} \). In this way we obtain a 4-dimensional QFT \( T_4[g,C] \). Its effective physics depends on the choice of a point of the Hitchin base \( \mathcal{B}(C,g) \):\(^{15}\)

**Definition 3.1.2** (Hitchin base). Fix a Riemann surface \( C \) and a Lie algebra \( g = \mathfrak{sl}(N) \). The *Hitchin base* \( \mathcal{B}(C,g) \) is

\[
\mathcal{B}(C,g) = \bigoplus_{n=2}^{N} H^0(C, \mathcal{K}_C^n).
\]

So a point of \( \mathcal{B}(C,g) \) is a tuple \( (\phi_2, \ldots, \phi_N) \) where \( \phi_n \) is a holomorphic \( n \)-differential on \( C \) (locally \( \phi_n = f(z)dz^n \)).

Next we place a surface defect on the subspace \( \{z\} \times \mathbb{R}^{1,1} \), taking \( R \) to be the fundamental (\( N \)-dimensional) representation of \( g \). In this way we obtain a 2-dimensional surface defect in \( T_4[g,C] \). We could call the combined system \( T_2[g,C,z] \).

\(^{15}\)In physics parlance \( \mathcal{B}(C,g) \) is the “Coulomb branch” of the theory \( T_4[g,C] \).
Indeed, as far as the supersymmetry algebra goes, this system behaves just like an ordinary $N = (2, 2)$ supersymmetric theory in $d = 2$: the ISO$(3, 1)$ symmetry is reduced to ISO$(1, 1)$, and the 8 supercharges are reduced to 4. So now we have realized $C$ as a moduli space of $N = (2, 2)$ supersymmetric theories.

**Physics Fact 3.1.4** (Chiral ring for surface defect in class $S$ theory). Fix $C, g, u \in \mathcal{B}(C, g)$ and $z \in C$. The chiral ring $R_z$ is isomorphic to $\mathbb{C}[\sigma]/(\sigma^N + \phi_1 \sigma^{N-1} + \cdots + \phi_N)$. The map $q : T_z C \to R_z$ takes $\partial_z \mapsto \sigma$.

As before, $q$ induces a Higgs field $\varphi$, so that $R$ is naturally a Higgs bundle. As a holomorphic bundle, $R$ is $O \oplus K_C \oplus K_C^2 \oplus \cdots \oplus K_C^{N-1}$. The Higgs bundle we want to consider is slightly different: we make a global twist $E = R \otimes K_C^{-2} = K_C^{-N} \oplus \cdots \oplus K_C^{N-1}$. Then $E$ is an $\mathfrak{s}(N)$-Higgs bundle, in fact lying in the “Hitchin section” from Schaposnik’s lectures.

The Hilbert space of $T_2[g, C, z]$ is somewhat different from the Landau-Ginzburg models we studied above, owing to the fact that the theory is “really” four-dimensional; as a result we will meet new phenomena.

**3.2. Spectral networks** The key player, as before, is a spectral network $SN(\emptyset)$ drawn on $C$, which keeps track of the counts of BPS particles living on the defect (which we still call “solitons.”)

Unlike the Landau-Ginzburg model, here we won’t give a direct construction of the solitons in terms of solving some equation along the string. Still, we can compute the spectrum using a spectral network.

**Definition 3.2.1** (Spectral network for class $S$ theory). Fix $C, g = \mathfrak{g}(N)$ or $\mathfrak{s}(N)$, and $u \in \mathcal{B}(C, g)$. Label the sheets of the spectral cover $\Sigma_u \to C$ by a (locally defined) index $i = 1, \ldots, N$; they correspond to holomorphic 1-forms $\lambda_i$ on $C$. 
Define an \(ij\)-trajectory with phase \(\vartheta\) on \(C\) to be a path along which

\[
e^{-i \vartheta} \int \lambda_i - \lambda_j \in \mathbb{R}.
\]

Locally the \(ij\)-trajectories give a foliation of \(C\).

Assume that the spectral curve \(\Sigma_u\) has only simple branch points. Then, the spectral network \(\text{SN}(\vartheta)\) on \(C\) is constructed by the following algorithm. Around each branch point, the foliation by \(ij\)-trajectories has a 3-pronged singularity.

We “shoot” three \(ij\)-trajectories from each \(ij\)-branch point, and evolve them following the differential equation (3.2.2). When an \(ij\)-trajectory crosses a \(jk\)-trajectory we create a new \(ik\)-trajectory born from the intersection point, and include it in the network as well.

If \(\Sigma_u\) has more interesting branching then the construction of \(\text{SN}(\vartheta)\) becomes more elaborate: we need to somehow determine the local structure around each branch point. Once we know that, we can proceed as above.

**Remark 3.2.3** (Spectral networks in the \(A_1\) case). When \(g = A_1\), all this becomes simpler:

- The data of \(u\) reduces to a holomorphic quadratic differential \(\phi_2\).
- The branch points of \(\Sigma_u\) are the zeroes of \(\phi_2\).
- The \(ij\)-trajectories are paths along which \(e^{-i \vartheta} \sqrt{\phi_2}\) is real. These are also called \(\vartheta\)-\textit{trajectories} of \(\phi_2\). These are well studied [Strebel, ...]
- \(\text{SN}(\vartheta)\) is also known as the \(\vartheta\)-\textit{critical graph}.

**3.3. A first example** Applying our definition of \(\text{SN}(\vartheta)\) we see a problem: a typical \(ij\)-trajectory will wind around the surface endlessly, with dense image, so \(\text{SN}(\vartheta)\) will be dense on \(C\). It might be possible to deal with that [ref Fenyes] but it’s surely not easy. We therefore add one extra ingredient: we allow \(C\) to have \textit{punctures}, and make the differentials \(\phi_n\) \textit{meromorphic} instead of holomorphic. Then the punctures can “attract” the trajectories, making the picture simpler. Here is a first example:
Example 3.3.1 (Argyres-Douglas theories of type $(A_1, A_2)$). We consider the case $C = \mathbb{CP}^1$, $g = sl(2)$, and

\[(3.3.2) \quad \phi_2 = (z^3 + \Lambda z + u) \, dz^2.\]

If we fix $\Lambda = 1$ and $u = 1$, some spectral networks $SN(\theta)$ are:

Remark 3.3.3 (Normalizable and non-normalizable modes). The parameters $\Lambda$ and $u$ above play rather different roles. $u$ should be thought of as an honest coordinate on the Hitchin base, while $\Lambda$ should be thought of as like a complex structure modulus associated with the puncture (despite the fact that naively a once-punctured $\mathbb{CP}^1$ has no moduli). Concretely, for each fixed $\Lambda$ there is expected to be a hyperkähler moduli space of Higgs bundles $M_\Lambda$, fibered over the $u$-plane; but the $M_\Lambda$ aren’t expected to fit together into a bigger hyperkähler space.

3.4. DT invariants in the $sl_2$ case

Comparing the pictures of spectral networks above, you notice something: the topology of $SN(\theta)$ abruptly changes at certain critical phases. At each critical phase there appears a trajectory connecting two branch points:

Definition 3.4.1 (Saddle connection). Given a holomorphic quadratic differential $\phi_2$, a saddle connection of $\phi_2$ is a path on $C$ which is a $\theta$-trajectory of $\phi_2$ for some $\theta$, with both ends on zeroes of $\phi_2$.

Definition 3.4.2 (Ring domain). Given a holomorphic quadratic differential $\phi_2$, a ring domain of $\phi_2$ is an annulus on $C$ foliated by $\theta$-trajectories of $\phi_2$ for some $\theta$.

Saddle connections and ring domains are well-studied objects in the Teichmüller theory community [refs].
Remark 3.4.3 (2d-4d decays). In terms of the physics of the surface defect, the topology change in $\text{SN}(\theta)$ is related to a process where some BPS particle states on the defect suddenly disappear. This process is hard to interpret in purely two-dimensional terms: it is not a decay of one soliton into other solitons. Rather, it is a process where the soliton decays into another soliton plus a particle in the four-dimensional bulk!

In other words, the saddle connection is a new kind of BPS particle: it is a BPS particle in the ambient 4-dimensional theory $T_4[g, C]$! This particular one turns out to be a massive hypermultiplet as in (1.6.27).

Physics Fact 3.4.4 (Electromagnetic charge lattice). Let
\[(3.4.5) \Gamma = H_1(\Sigma_u, \mathbb{Z}).\]
$\Gamma$ is the lattice of electromagnetic charges in the theory $T_4[g]$ at the point $u$ of its Coulomb branch. (This is precisely true for $g = \mathfrak{gl}_N$, a slight lie for $g = \mathfrak{sl}_N$.)

Definition 3.4.6 (Charge of a saddle connection or ring domain). For each saddle connection one can define a class $\gamma \in \Gamma$, represented by the lift to $\Sigma_u$ of a loop around the branch points, as shown in the above figure. [explain how to fix orientation] We call $\gamma$ the charge of the saddle connection. Likewise for each ring domain of $\phi_2$ there is a charge $\gamma \in \Gamma$ given by the sum of the two lifts to $\Sigma_u$.

Definition 3.4.7 (Donaldson-Thomas invariants when $g = \mathfrak{sl}_2$). For a fixed $g = \mathfrak{sl}_2$, $C$ and $u = \phi_2$ we define
\[(3.4.8) \Omega(\gamma) = (\# \text{saddle connections with charge } \gamma) - 2(\# \text{closed loops with charge } \gamma).\]

Physics Fact 3.4.9 (Physics meaning of the DT invariants). $\Omega(\gamma)$ is a BPS index counting 1-particle states of electromagnetic charge $\gamma$ in the theory $T_4[g]$, in the sense of (1.6.26).

3.5. DT invariants in higher rank cases

Example 3.5.1 (Argyres-Douglas theories of type $(A_2, A_1)$). We consider the case $C = \mathbb{CP}^1$, $g = \mathfrak{sl}(3)$, and
\[(3.5.2) \phi_2 = \Lambda \, dz^2, \quad \phi_3 = (z^2 + u) \, dz^3.\]
Here is a picture at $u = 1, \Lambda = 1$:
In this case, varying the phase ϑ we again meet a topology change, but it is not associated with a saddle connection anymore: [...]  

3.6. Families of flat connections  Just as in the pure 2d case, the Higgs bundle E over C carries a Hermitian metric solving the tt* equations aka Hitchin equations. Thus one has a family of flat connections ∇ζ (ζ ∈ C^n) in E, given by (2.5.7).

There is also a family ∇h of flat connections (“opers”) which are the analogue of the topological connections above. Those we can actually write explicitly. For the AD theories above, we would have

\[ ∇h = ∂ + \begin{pmatrix} 0 & p_2 \\ 1 & 0 \end{pmatrix} dz. \]

In this case the ∇h-flatness equation is equivalent to a Schrödinger equation, written as [signs?]

\[ (\hbar^2 \partial_z^2 + p(z))\psi(z) = 0. \]

The spectral network is a useful tool for studying both of these families. In each case the strategy is the same.

**Conjecture 3.6.3.** There exist flat sections [...]  

[distinguished solutions, abelianization]  
[definition of \( X_γ \)]  
[continuity of \( X_γ \) except at critical phases]

**References**

References

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