A wall-crossing formula for 2d-4d DT invariants

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Preface

In the last few years there has been a lot of progress in the theory of generalized Donaldson-Thomas invariants.

[Kontsevich-Soibelman, Joyce-Song, ...]

In many (all?) cases where they can be defined, these invariants have a physical meaning: BPS state index in 4d $\mathcal{N} = 2$ SUSY quantum systems.

[Ooguri-Strominger-Vafa, Denef, Denef-Moore, Gaiotto-Moore-N, Dimofte-Gukov-Soibelman, Cecotti-Vafa, ...]

Preface

This talk is motivated by some new progress in physics, involving new "2d-4d" BPS state indices, which can be defined when we add a 2d surface operator to our 4d quantum system.

We have learned a few basic facts about these 2d-4d indices — in particular we have learned what their wall-crossing formula is. The main aim of this talk is to explain these facts.

We conjecture that the 2d-4d indices should be part of a not-yet-formulated extension of Donaldson-Thomas theory.

These 2d-4d indices seem to be useful tools even if your ultimate interest is only in the original generalized DT invariants.

Preface

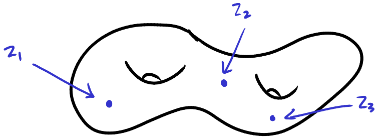
I'll focus on a specific class of situations where both the generalized DT invariants and the new 2d-4d invariants can be defined relatively easily, and the new 2d-4d wall-crossing formula can be directly checked.

In these examples, the generalized DT invariants (as well as their 2d-4d extensions) are counting special trajectories associated with a quadratic differential on a Riemann surface C.

First, I'll review the "old" story; then I'll give its 2d-4d extension.

Invariants of a quadratic differential

Fix compact Riemann surface C, with $\ell > 0$ marked points z_i, $i = 1, \ldots, \ell$. Let C' be C with the marked points deleted. Fix (generic) parameters $m_i \in \mathbb{C}$ for each i.



Let $\mathcal{B}$ be the space of meromorphic quadratic differentials φ_2 on C, with double pole at each z_i, residue m_i^2:

$$\varphi_2 = \frac{m_i^2}{(z - z_i)^2} dz^2 + \cdots$$

Invariants of a quadratic differential

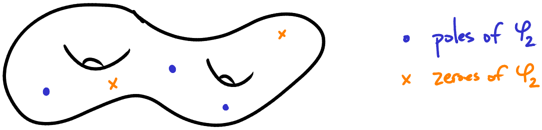
Fix a point of $\mathcal{B}$, i.e. fix a meromorphic quadratic differential φ_2 on C with double pole at each z_i, residue m_i.

This determines a metric h on C, in a simple way:

$$h = |\varphi_2|$$

(so if $\varphi_2 = P(z)\,dz^2$ then $h = |P(z)|\,dz\,d\bar{z}$.)

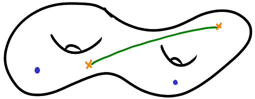
More precisely, h is a metric on only an open subset of C, where we delete both the poles of φ_2 (the z_i) and also the zeroes of φ_2. h is flat on this open subset.



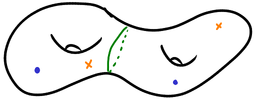
Invariants of a quadratic differential

Now we can consider finite length inextendible geodesics on C' in the metric h. These come in two types:

- Saddle connections: geodesics running between two zeroes of φ_2. These are rigid (don't come in families).



- Closed geodesics. When they exist, these come in 1-parameter families, sweeping out annuli on C'.

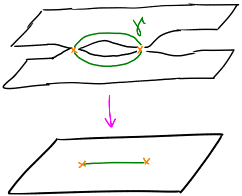


Invariants of a quadratic differential

To "classify" these finite length geodesics, introduce a little more technology: φ_2 determines a branched double cover $\Sigma \to C'$,

$$\Sigma = \{x : x^2 = \varphi_2\} \subset T^*C.$$

Each finite length geodesic can be lifted to a union of closed curves in Σ, representing some homology class $\gamma \in H_1(\Sigma, \mathbb{Z})^{odd}$.

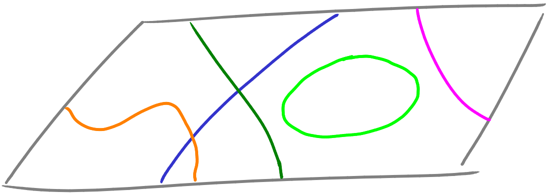


We define an invariant

$$\Omega(\gamma) = \begin{cases} 1 & \text{if there is a saddle connection w/ lift } \gamma \\ -2 & \text{if there is a closed geodesic w/ lift } \gamma \\ 0 & \text{if neither} \end{cases}$$

Walls

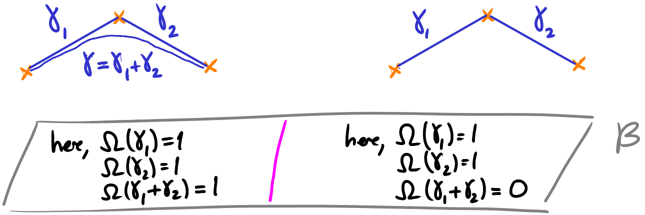
As we vary φ_2, $\Omega(\gamma)$ can jump, when some finite-length geodesics appear or disappear. This occurs at some real-codimension-1 loci $W \subset \mathcal{B}$ ("walls").



Walls

As we vary φ_2, $\Omega(\gamma)$ can jump, when some finite-length geodesics appear or disappear.

Basic mechanism: decay/formation of bound states.



Where is the wall? Define a function Z_γ on $\mathcal{B}$ ($\mathcal{B}$ is the parameter-space of quadratic differentials φ_2) by

$$Z_\gamma = \oint_\gamma \lambda$$

where λ is the canonical 1-form on T^*C. Then the wall is the locus in $\mathcal{B}$ where

$$Z_{\gamma_1}/Z_{\gamma_2} \in \mathbb{R}.$$

Wall-crossing formula

The jump of the $\Omega(\gamma)$ at the wall is governed by the Kontsevich-Soibelman WCF.

To state that formula (which will take a few slides), we axiomatize our structure a bit: the data are

- Complex manifold $\mathcal{B}$ (space of quadratic differentials on C)
- Lattice Γ w/ antisymmetric pairing $\langle , \rangle$ ($H_1(\Sigma, \mathbb{Z})^{odd}$)
- Homomorphism $Z : \Gamma \to \mathbb{C}$ for each point of $\mathcal{B}$, varying holomorphically over $\mathcal{B}$ ($Z_\gamma = \oint_\gamma \lambda$)
- "invariants" $\Omega : \Gamma \to \mathbb{Z}$ for each point of $\mathcal{B}$ (counts of finite length geodesics)

Wall-crossing formula

Attach a "$\mathcal{K}$-ray" in $\mathbb{C}$ to each γ with $\Omega(\gamma) \neq 0$.

Slope of the $\mathcal{K}$-ray is given by the argument of Z_γ.

These rays move around as we vary the quadratic differential φ_2, i.e. as we move in $\mathcal{B}$.

Walls in $\mathcal{B}$ are loci where some set of rays become aligned:

Near the wall. On the wall.

Focus on these participating rays.

Wall-crossing formula

Introduce torus $T \simeq (\mathbb{C}^\times)^{\mathrm{rank}\ \Gamma}$ with coordinate functions $X_\gamma : T \to \mathbb{C}^\times$ for each $\gamma \in \Gamma$, obeying

$$X_\gamma X_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} X_{\gamma+\gamma'}.$$

To each γ, assign a (birational) automorphism $\mathcal{K}_\gamma$ of T:

$$\mathcal{K}_\gamma : X_{\gamma'} \mapsto (1 - X_\gamma)^{\langle \gamma, \gamma' \rangle} X_{\gamma'}$$

Now consider a product over all participating γ,

$$: \prod_\gamma \mathcal{K}_\gamma^{\Omega(\gamma)} :$$

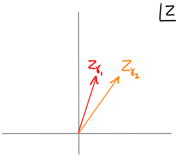
where :: means we multiply in order of the phase of Z_γ.

The Kontsevich-Soibelman WCF is the statement that this automorphism is the same on both sides of the wall.

Wall-crossing formula

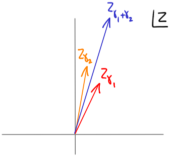
For example: if $\langle \gamma_1, \gamma_2 \rangle = 1$,

$$\mathcal{K}_{\gamma_1} \mathcal{K}_{\gamma_2}$$



equals

$$\mathcal{K}_{\gamma_2}^{\Omega'(\gamma_2)} \mathcal{K}_{\gamma_1+\gamma_2}^{\Omega'(\gamma_1+\gamma_2)} \mathcal{K}_{\gamma_1}^{\Omega'(\gamma_1)}$$



if and only if

$$\Omega'(\gamma_1) = 1$$
$$\Omega'(\gamma_2) = 1$$
$$\Omega'(\gamma_1 + \gamma_2) = 1$$

Wall-crossing formula

More interesting example: if $\langle \gamma_1, \gamma_2 \rangle = 2$,

$$\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_2} = \left(\prod_{n=0}^{\infty} \mathcal{K}_{n\gamma_1+(n+1)\gamma_2}\right)\mathcal{K}_{\gamma_1+\gamma_2}^{-2}\left(\prod_{n=\infty}^{0} \mathcal{K}_{(n+1)\gamma_1+n\gamma_2}\right)$$

So,

- on one side of the wall we have only $\Omega(\gamma_1) = 1$ and $\Omega(\gamma_2) = 1$, all others zero;
- on the other side we have infinitely many $\Omega(\gamma) = 1$, and also $\Omega(\gamma_1 + \gamma_2) = -2$.

Wall-crossing formula

Key fact: KS WCF holds for our integer invariants $\Omega(\gamma)$!

So e.g.

$$\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_2} = \mathcal{K}_{\gamma_2}\mathcal{K}_{\gamma_1+\gamma_2}\mathcal{K}_{\gamma_1}$$

for $\langle\gamma_1,\gamma_2\rangle = 1$ says that if we have two saddle connections that intersect at 1 point, then after wall-crossing a third saddle connection will appear.

Similarly in the formula

$$\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_2} = \left(\prod_{n=0}^{\infty}\mathcal{K}_{n\gamma_1+(n+1)\gamma_2}\right)\mathcal{K}_{\gamma_1+\gamma_2}^{-2}\left(\prod_{n=\infty}^{0}\mathcal{K}_{(n+1)\gamma_1+n\gamma_2}\right)$$

for $\langle\gamma_1,\gamma_2\rangle = 2$, on one side we have two saddle connections intersecting at two points; on the other side we have infinitely many saddle connections plus a single closed geodesic.

Segue

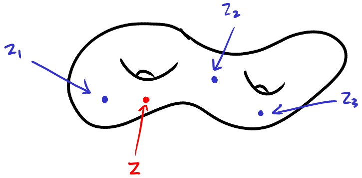
So far, so good: we described a simple class of enumerative invariants $\Omega(\gamma) \in \mathbb{Z}$ attached to a curve C with some marked points, and explained that they give examples of the general wall-crossing formula of Kontsevich-Soibelman.

Now, we consider our "2d-4d" extension. $\Omega(\gamma) \in \mathbb{Z}$ will be slightly refined to some new objects $\omega(\gamma, \gamma_{ij}) \in \mathbb{Z}$, and we will also introduce new $\mu(\gamma_{ij}) \in \mathbb{Z}$.

Invariants of a quadratic differential plus a point

As before: Fix compact Riemann surface C, with $n > 0$ marked points z_i, $i = 1, \ldots, n$. Let C' be C with the marked points deleted. Fix (generic) parameters $m_i \in \mathbb{C}$ for each i.

New 2d-4d datum: Fix a point $z \in C$.



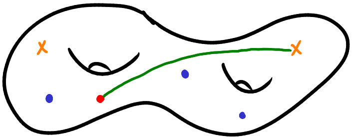
As before: let $\mathcal{B}$ be the space of meromorphic quadratic differentials φ_2 on C, with double pole at each z_i, residue m_i^2:

$$\varphi_2 = \frac{m_i^2}{(z - z_i)^2} dz^2 + \cdots$$

Invariants of a quadratic differential plus a point

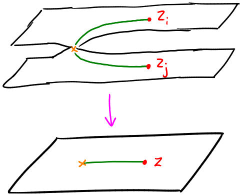
As before: we are interested in counting finite-length geodesics on C', in the flat metric h determined by φ_2.

However, now we allow them to be open, i.e. to have one end on the point z. (And the other end on a zero of φ_2 as before.)



Invariants of a quadratic differential plus a point

To categorize these open geodesics, we again consider their lifts to the double cover Σ:



These give 1-chains γ_{ij} with $\partial \gamma_{ij} = z_i - z_j$; let Γ_{ij} be set of such 1-chains modulo boundaries.

Γ_{ij} is a torsor over the homology Γ. For any $\gamma_{ij} \in \Gamma_{ij}$, we define new invariant $\mu(\gamma_{ij})$ by

$$\mu(\gamma_{ij}) = \begin{cases} 1 & \text{if there is an open geodesic w/ lift } \gamma_{ij} \\ 0 & \text{if not} \end{cases}$$

Invariants of a quadratic differential plus a point

In the presence of the extra point z, we can also keep track of slightly more information about the ordinary finite geodesics: measure their homology classes on Σ punctured at the preimages of z.

To encode that information: for any $\gamma \in \Gamma$ and $\gamma_{ij} \in \Gamma_{ij}$, we define new invariant $\omega(\gamma, \gamma_{ij})$, by

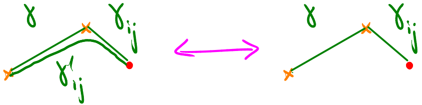
$$\omega(\gamma, \gamma_{ij}) = \Omega(\gamma)\langle \gamma, \gamma_{ij} \rangle$$

(To define the intersection number with the open path γ_{ij}, if γ is an isolated geodesic, use the actual geodesic representative for γ. If γ not isolated, use the two ends of the family, take the average.)

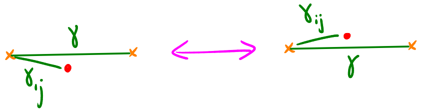
Walls

As before: as we vary φ_2 and z, $\mu(\gamma)$ and $\omega(\gamma, \gamma_{ij})$ can jump.

Two sample pictures:



$\mu(\gamma_{ij})$ jumps



$\omega(\gamma, \gamma_{ij})$ jumps

2d-4d wall-crossing formula

To state our extended "2d-4d" WCF, axiomatize our new structure: to our old list

- Complex manifold $\mathcal{B}$ (space of quadratic differentials on C)
- Lattice Γ w/ antisymmetric pairing $\langle , \rangle$ ($H^1(\Sigma, \mathbb{Z})^{odd}$)
- Homomorphism $Z : \Gamma \to \mathbb{C}$ for each point of $\mathcal{B}$, varying holomorphically over $\mathcal{B}$ ($Z_\gamma = \oint_\gamma \lambda$)
- "invariants" $\Omega : \Gamma \to \mathbb{Z}$ for each point of $\mathcal{B}$ (counts of finite geodesics)

we now add

- Γ-torsors Γ_{ij} for $i, j = 1, \ldots, n$, with addition operations $\Gamma_{ij} \times \Gamma_{jk} \to \Gamma_{ik}$, satisfying associativity ($n = 2$, spaces of 1-chains with boundary $z_i - z_j$)
- Maps $Z : \Gamma_{ij} \to \mathbb{C}$ obeying additivity ($Z_{\gamma_{ij}} = \int_{\gamma_{ij}} \lambda$)
- "invariants" $\omega : \Gamma \times \Gamma_{ij} \to \mathbb{Z}$, satisfying $\omega(\gamma, \gamma' + \gamma_{ij}) = \omega(\gamma, \gamma_{ij}) + \Omega(\gamma)\langle\gamma, \gamma'\rangle$ (refined counts of finite geodesics)
- "invariants" $\mu : \Gamma_{ij} \to \mathbb{Z}$ for each i, j with $i \neq j$, and each point of $\mathcal{B}$ (counts of open geodesics)

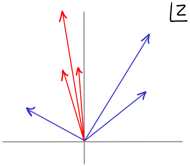
2d-4d wall-crossing formula

As before, attach a ray in $\mathbb{C}$ to each nonzero invariant: "$\mathcal{K}$-ray" for each γ with $\omega(\gamma, \cdot) \neq 0$, "$\mathcal{S}$-ray" for each γ_{ij} with $\mu(\gamma_{ij}) \neq 0$.

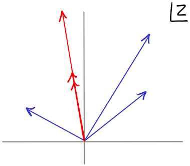
Slope of the rays given by the argument of Z_γ or $Z_{\gamma_{ij}}$.

The rays move around as we vary the quadratic differential φ_2 and the point z, i.e. as we move in $\mathcal{B} \times C$.

As before, walls in $\mathcal{B} \times C$ are loci where some set of rays become aligned:



Near the wall.



On the wall.

Focus on these participating rays.

2d-4d wall-crossing formula

To formulate the KS WCF, we used an auxiliary gadget, the torus $T \simeq (\mathbb{C}^\times)^{\mathrm{rank}\,\Gamma}$. The $\Omega(\gamma)$ got encoded into automorphisms $\mathcal{K}_\gamma^{\Omega(\gamma)}$ of T.

For the 2d-4d WCF, we decorate that story a bit: add a trivializable holomorphic rank-n vector bundle V over T. The 2d-4d invariants get encoded into automorphisms of that object:

- The $\omega(\gamma, \cdot)$ contain slightly more information than $\Omega(\gamma)$; correspondingly, they determine an object $\mathcal{K}_\gamma^\omega$, which lifts $\mathcal{K}_\gamma^{\Omega(\gamma)}$ to act on V.
- The new invariants $\mu(\gamma_{ij})$ give new automorphisms $\mathcal{S}_{\gamma_{ij}}^{\mu(\gamma_{ij})}$ which leave points of T fixed, act only on the fiber of V. (Unipotent matrices with one off-diagonal entry, in the ij place.)

2d-4d wall-crossing formula

Just to show you that the formulas are concrete:

$\mathcal{K}_\gamma^\omega$ and $\mathcal{S}_{\gamma_{ij}}$ are automorphisms of a vector bundle V over T. I'll give their action on a basis of sections of $End(V)$: so along with the X_γ we had before (functions on T), now also have sections $X_{\gamma_{ij}}$ ("elementary matrix" sections of $End(V)$), and the automorphisms act by

$$\mathcal{K}_\gamma^\omega : X_{\gamma_{ij}} \mapsto (1 - X_\gamma)^{\omega(\gamma, \gamma_{ij})} X_{\gamma_{ij}}$$
$$\mathcal{S}_{\gamma_{kl}}^\mu : X_{\gamma_{ij}} \mapsto (1 + \mu X_{\gamma_{kl}}) X_{\gamma_{ij}} (1 - \mu X_{\gamma_{kl}})$$

This is enough for our purposes.

Wall-crossing formula

Now consider a product over all participating rays

$$: \prod_{\gamma, \gamma_{ij}} \mathcal{K}_{\gamma}^{\omega} \mathcal{S}_{\gamma_{ij}}^{\mu} :$$

where :: means we multiply in order of the phase of Z_γ, $Z_{\gamma_{ij}}$.

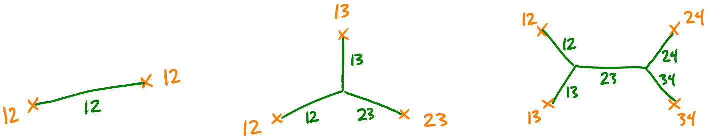
This object is an automorphism of the torus T, lifted to act on the vector bundle V.

The 2d-4d WCF is the statement that this automorphism is the same on both sides of the wall.

Networks

Story so far might seem a little lame: 2d-4d WCF involves an integer n, but my only example of 2d-4d invariants had $n = 2$.

There is a more general version: the quadratic differential φ_2 is replaced by a tuple of k-differentials $(\varphi_k)_{2 \leq k \leq n}$, and instead of counting geodesics on C, we count certain networks:

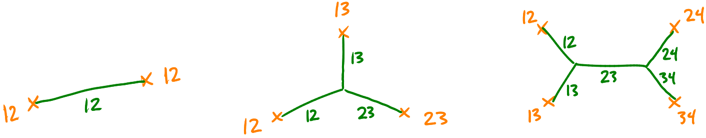


Each leg is labeled by a pair of sheets (x_i, x_j) of the n-fold covering

$$\Sigma = \{x^n + \sum_{k=0}^{n-2} x^k \varphi_{n-k} = 0\} \subset T^*C$$

and is a straight line in the coordinate $\int x_i - x_j$, with inclination ϑ, the same for all legs.

Networks



We don't know anywhere that these networks have been studied before (would be very curious to learn a reference!)

They are a natural generalization of the special trajectories of quadratic differentials, which are better-studied.

[Strebel, Hubbard, Masur, Kontsevich, Zorich, ...]

We predict that the counting of these objects — and their open analogues, with one leg ending on a marked point z — is governed by our 2d-4d WCF (work in progress).

2d-4d wall-crossing formula

The 2d-4d WCF does not come out of nowhere.

In 2d field theories there is a WCF with structure similar to the KS WCF and the 2d-4d WCF. [Cecotti-Vafa]

In that WCF, the relevant group of "automorphisms" which appears is just $GL(n)$.

The 2d-4d WCF is a kind of hybrid between that WCF and the 4d KS WCF.

(It fits into a general formalism where the "invariants" are allowed to belong to a more general graded Lie algebra.) [Kontsevich-Soibelman]

Applications

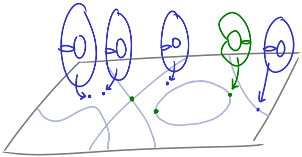
What are the 2d-4d invariants good for?

If you are in a situation where the 2d-4d invariants can be defined, and you have complete knowledge of μ, you can get information about the original generalized DT invariants Ω. (Because the wall-crossing formula connects the two.)

Indeed in many cases μ actually determines Ω! And μ seem to be somewhat more easily computable at least in some examples...

A geometric application

One application of the KS WCF: a construction of a family of hyperkähler metrics on the total space $\mathcal{M}$ of a torus bundle over $\mathcal{B}$.



Very roughly, the idea is to build a $\mathbb{C}^\times$ worth of complex symplectic structures on $\mathcal{M}$, then show these structures are induced from the desired hyperkähler metric.

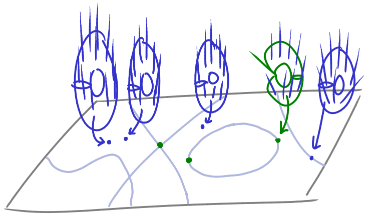
Construction proceeds by gluing together patches which look like the torus T, with the gluing maps given by the automorphisms $\mathcal{K}_\gamma^{\Omega(\gamma)}$.

[Strominger-Yau-Zaslow, Kontsevich-Soibelman, Auroux, Gross-Siebert, ...]

In the examples I have been discussing here, $\mathcal{M}$ is a moduli space of solutions of Hitchin equations over C with gauge group $SU(n)$.

A geometric application

The 2d-4d WCF gives a new construction of a hyperholomorphic bundle $\mathcal{V}$ over this hyperkähler $\mathcal{M}$ (think of this as a generalization of a Yang-Mills instanton.) Very roughly, construction proceeds by gluing together patches which look like the vector bundle V over T, with gluing maps given by $\mathcal{K}^{\omega}_{\gamma}$ and $\mathcal{S}^{\mu_{ij}}_{\gamma_{ij}}$.



In the examples I have been discussing here, $\mathcal{M}$ is a moduli space of solutions of Hitchin equations over C with gauge group $SU(k)$, $\mathcal{V}$ is the universal bundle restricted to $z \in C$.

2d-4d invariants in general

We conjecture that 2d-4d invariants obeying the 2d-4d WCF exist in other "Donaldson-Thomas situations" as well.

Roughly speaking, whenever we have a (A or B) brane whose moduli space is 0-dimensional, we should be able to use it to define a 2d-4d version of the counting of (B or A) branes.

(A funny-looking mixing of the two mirror-dual categories!)

2d-4d invariants for interpolating cycles

For example, suppose we have a Calabi-Yau threefold X. The standard dogma is that there should be generalized Donaldson-Thomas invariants $\Omega(\gamma)$ which "count" special Lagrangian cycles (A branes) on X, for $\gamma \in \Gamma = H_3(X, \mathbb{Z})$.

Now fix a class in $H_2(X, \mathbb{Z})$, with finitely many representative holomorphic curves $Y_1, \dots, Y_n$ (B branes).

Let Γ_{ij} be the set of 3-chains with boundary $Y_i - Y_j$, for any $i, j = 1, \dots, n$. Each Γ_{ij} is a torsor over Γ.

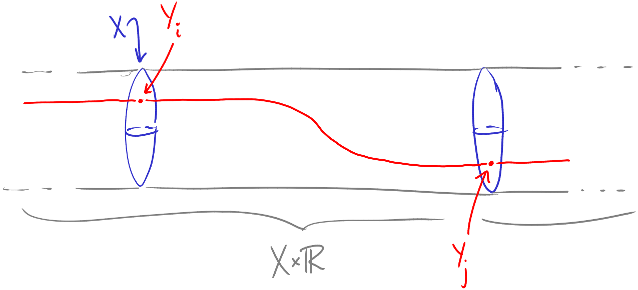
In this situation, with luck, there should be 2d-4d invariants $\omega(\gamma, \gamma_{ij})$ and $\mu(\gamma_{ij})$, with $i, j = 1, \dots, n$, and $\gamma_{ij} \in \Gamma_{ij}$.

2d-4d invariants for interpolating cycles

The physics of the situation suggests one possible picture of what the 2d-4d invariants $\mu(\gamma_{ij})$ are counting in this case:

Consider the 7-manifold $X \times \mathbb{R}$, with holonomy $SU(3) \subset G_2$.

The 3-cycles $Y_i \times \mathbb{R}$ and $Y_j \times \mathbb{R}$ are associative cycles (calibrated by the G_2 3-form). $\mu(\gamma_{ij})$ should be counting associative cycles which interpolate between these two.



Conclusions

Motivated by physics, we propose that there should exist a new "2d-4d" extension of the usual theory of generalized Donaldson-Thomas invariants. $\Omega(\gamma)$ replaced by $\mu(\gamma_{ij})$ and $\omega(\gamma, \gamma_{ij})$.

In particular, we claim this new theory governs the counting of certain open and closed networks of trajectories on Riemann surfaces.

While we don't know how to define this theory in general, we do know what its wall-crossing behavior should be.