1. Recollection on nonabelian Hodge

C compact Riemann surface
\( G = \text{SU}(k) \)

Higgs bundle: \( (E, \phi) \) \( E \) hermit. rank \( k \), over \( C \)
\( \phi \in \text{H}^0(End E \otimes \Omega^2) \)

Given a (polystable) Higgs bundle, \( \exists \) a harmonic metric \( h \) in \( E \) s.t.
\( \frac{\partial}{\partial S} \left[ \frac{S}{h} + D_k + \phi^* S \right] = 0 \) \( (D_k = \text{Chen connection in } E, \text{ unitary}) \) [Hitchin, Simpson]

Then for any \( S \in \mathbb{C} \), \( \nabla_S = \nabla + D_k + \phi^* S \) is a flat complex connection

For any fixed \( S \in \mathbb{C} \), the map \( (E, \phi) \mapsto \nabla_S \) induces a diffeomorphism of moduli spaces, "nonabelian Hodge map" (NAH)

The NAH induce a hyperkähler structure on \( M_{\text{Higgs}} \).

In particular, a metric, whose Kähler form is \( \omega = \text{Re} \left( \Phi^1 \Phi^2 \right) \), [cf. Zhang talk]

Q: What can we understand explicitly?

2. Some baby examples

Find an analogue of this theory where \( \Psi \) is allowed poles, subject to some conditions.

Fix \( N \), then \( \exists \) a space \( M_{K_N} \) of Higgs bundles over \( \mathbb{CP}^1 \), where eigenvalues of \( \Psi \) have

\( \lambda \sim \frac{N}{2} \frac{d^2}{dz^2} \) [w/Laura Fredrickson]

Inside \( M_{K_N} \), there's a half-dimensional slice, "Hitchin component"

\( K = 2 \) \( \{ E = O_{\mathbb{C}^1}^{(N)} \oplus O_{\mathbb{C}^1}^{(-N)}, \Psi = \begin{pmatrix} 0 & 1 \\ P_2(z) & 0 \end{pmatrix} \frac{d^2}{dz^2} \text{ with } P_2(z) = z^N + (\text{degree } \leq \frac{N-1}{2}) \}\)

\( \text{NAH at } S = \frac{1}{2} \)

\( \frac{1}{2} \) harmonic maps \( F: \mathbb{C} \rightarrow \text{SL}_2 \mathbb{R}/\text{SO}_2 \) with image ideal \( (N+2)g \) [Hua--Tam--Fredericks--Wed] [Wolf]

\( (Q, \text{Li talk}) \)
$K = 3$: \( \{ \varepsilon = O(3) \oplus O(3) \} \), \( \varphi = \begin{pmatrix} 0 & 0 \\ \frac{P_2(z)}{P_3(z)} & 0 \end{pmatrix} \) with \( P_2(z) = (\text{degree} \leq \frac{N-1}{3}) \), \( P_3(z) = z^N + (\text{degree} \leq \frac{2N-1}{3}) \)

\[ \text{coeff of } P_2, P_3 \]

NAH at \( f = 5 \)!

- \{ \text{harmonic maps } F: C \to SL_3 \mathbb{R}/SO_3 \text{ with "polynomial growth"} \}

- \text{SL}_{3} \mathbb{R}

- [Laffon, Laboucin, Dumas-Wolf] when \( P_2 = 0 \): polynomial affine spheres

\( \text{cross ratios of } N+3 \text{ pts in } \mathbb{R}P^2 \) (convex polyg.)

\[ \text{Suppose } \prod_{k=1}^{N} \left( \frac{z^k}{w_k} \right)^{\beta_k} \]

\[ \text{Let } \{ X_A, X_B \} \text{ be cross ratios of the polygon: } X_A = r_{1235}, X_B = r_{1345} \quad (X_A = X_B) \]

Weak prediction: if we replace \( P_2 \to R^2 P_2 \) then \( X_A = \exp \left( -5.04 \pi \cdot R + 8(R) \right) \), \( \lim_{R \to 0} \delta(R) = 0 \)

\[ \text{Weak prediction: } \delta(R) = -\frac{3}{2N \pi R i} e^{-2 \pi i R} + \tilde{\delta}(R) \]

\[ \lim_{R \to 0} \sqrt{R} e^{2 \pi i R} \delta(R) = 0 \]

Strong prediction: \( X_A \) can be extended to a piecewise analytic function \( X_A(\xi), \xi \in \mathbb{C}^x \), obeying the integral eq:

\[ X_A(\xi) = \exp \left[ -i Z_A + 5 Z_A \right] + \int_{\mathbb{C}} \frac{1}{4 \pi i} \sum_{\mu \in \Gamma} \Omega(\mu) \chi_{\mu} \left( \frac{5}{8} \right) \frac{5}{8} \sum_{\xi \in Z_A} \log \left( 1 + \chi_{\mu}(\xi) \right) \]
Why study abelianization of the family of flat connections $\nabla^3$ and their asymptotics as $t \to 0$? "enjoy" Stokes phenomena controlled by the saddle connections; int eq. goes simplest real.

There are similar predictions for some $K=3$ examples, e.g., $P_3 = z^3 - 3z - 2$.

4) Numerical results

[Dumas-Wolf] code for computing the polygons in $K=3$ case, by directly solving PDE.

[Dumas-N] or integral equations.

e.g., for $K=2$, have to solve $\Delta u - 4e^{2u} + 4e^{-2u}|P_2|^2 = 0$.

Show results. Large $R$: best for $\int$ eq. Small $R$: best for direct PDE. (show example $R=2$)

5) Hyperkahler metrics

$\Omega = d\log X_A \wedge d\log X_B \Rightarrow$ use these formulas to predict the HK metric.

e.g., "weakest prediction" $\Rightarrow$ HK metric has $\|P_2\|^2 = \int \frac{|P_2|^2}{\Omega} \ "semiflat metric"$.

Show results.