Def For $S$ top. space, $D \subset S$,
\[
\pi_{\leq 1}(S \setminus D) = \text{fundamental groupoid of } (S \setminus D)
\]

This has: objects $D$,
morphisms $(x_1, x_2) = \{\text{paths from } x_1 \text{ to } x_2 \}/\text{homotopy}

\[
\begin{array}{ccc}
  a & \xrightarrow{b} & c \\
  \downarrow & & \downarrow \\
  a' & \xrightarrow{a'z} & a
\end{array}
\]

Now take $\sum \pi_1$ branched over $\Delta$, as before, and $z \in C$.

Def $\mathcal{A}(\Sigma, z) = \mathbb{Z}[\pi_{\leq 1}(\Sigma \setminus \pi^{-1}(\Delta), \pi^{-1}(z))]/\sim$

where $a_1 \sim a_2$ if there's a homotopy in $\Sigma$ from $a_1$ to $a_2$ crossing $\Delta$ once.

Remark: The rings $\mathcal{A}(\Sigma, z)$ form a local system of rings over $C \setminus \Delta$.

\[
\begin{array}{ccc}
  z & \xrightarrow{2} & z' \\
  \downarrow & & \downarrow \\
  z & \xrightarrow{2} & z'
\end{array}
\]

$\mathcal{A}(\Sigma, z) = \bigoplus_{i,j} \mathcal{A}_{ij}(\Sigma, z)$

Remark: Similarly define rings $\mathcal{A}(\Sigma, \{z_1, z_2, z_3\})$. 
Path lifting

1. Solitons

Each \( ij \)-trajectory \( p \in W = W(\Sigma, W) \) carries extra datum

\[ s_p(z) \in \Lambda(\Sigma, z) \otimes \mathbb{Z}_2 \]

determined inductively:

a) for \( p \) born from branch point,

\[ s_p(z) \text{ given by } \]

b) for \( p \) born from intersection of \( \bar{p}_a \) and \( \bar{p}_b \),

\[ s_p = s_{\bar{p}_a} s_{\bar{p}_b} \]

(both with cover \( \Upsilon \))

\[ \text{Fact: } \text{org}(s_p, \lambda) = 0. \]

(Thus can measure "length" of \( s_p(z) \) as \( \int_{\Sigma, z} \lambda \), monotonically increasing.)

2. Lifts

From now on assume \( W \) is finite.

For \( z, z' \in W \) define

\[ F = \mathbb{F}: \pi_{\leq 1}(\Sigma, \{z_1, z_2\}) \rightarrow A(\Sigma, \{z_1, z_2\}) \]

- For a path \( p \) not meeting \( W \)

\[ F(p) = \sum_{i=1}^{N} P^{(i)} \]

- For a path \( p \) meeting \( W \) once in interior

\[ F(p) = \sum_{i=1}^{N} P^{(i)} + P^{(j)} S_p(z) P^{(k)} \]

- For general \( p \) on \( C - \Delta \), ends at \( \varphi \) on \( W \), break into pieces and require

\[ F(p, \varphi) = F(p) F(\varphi) \]
Prop \( F \) is well defined.

Key part: Homotopy invariance

\[ \text{eq} \]

\[ F(P) = \]

\[ F(P') = \]

\[ F(P) = F(P') \]

Similarly

\[ F(P) \text{ includes} \]

\[ F(P') \text{ includes} \]

\[ F(P) = F(P') \]
Nonabelianization

"Dual" to $F$ is a map (functor) $\hat{F} : \widetilde{M}(\Sigma, GL(n)) = \text{[almost flat } GL(n)\text{-conn. over } \Sigma]\to \text{[flat } GL(n)\text{-conn over } C\text{ punctures]}$

Idea of $\hat{F}$: $\sqcup \triangledown \text{ flat } GL(n)\text{-conn. in } \Sigma \to \Sigma$

If $F(P) = \sum k \xi_k P_k$, $\hat{H}_0(\hat{F}(P)) = \sum k \xi_k \hat{H}_0(P_k)$