References

Gaiotto, Moore, N. "Four-dimensional wall crossing via three-dimensional field theory."
Wall-crossing, Hitchin systems, and the WKB approximation.
Spectral networks.

Gaiotto. "Operas and TBA."

Iwaki, Nakanishi. "Exact WKB analysis and cluster algebras."

Allegretti, Bridgeland. "The monodromy of meromorphic projective structures."

Hollands, N. "Exact WKB ad abelianization for the $T_3$ equation."
Integral iterations for harmonic maps.

Bridgeland, Smith. "Quadratic differentials as stability conditions."

Fock, Goncharov. "Moduli spaces of local systems and higher Teichmüller theory."

Kontsevich, Soibelman. "Stability structures, motivic Donaldson-Thomas invariants and cluster transformations."
Last time:
\[
\begin{align*}
[&C \text{ compact R.s.}] \\
&u = (\phi_1, \ldots, \phi_N) \\
&\sigma \in \mathbb{R}/2\pi\mathbb{Z}
\end{align*}
\] 

\[
\frac{1}{\tau} \text{ spectral curve} \quad \Sigma_u \xrightarrow{\pi} C \\
\text{ spectral network} \quad W = W(u, \sigma)
\]

\[
\text{ path lifting/"modularization"} \\
F_{wu} : \tilde{\mathcal{M}}(\Sigma_u, \mathbb{C}G(1)) \to \mathcal{M}(C, \mathbb{C}G(N))
\]

\[
\nabla^0 \mapsto \nabla
\]

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1. **Spectral coordinates and DT invariants**

Say we fix (generic) conjugacy classes of monodromy around poles of $u$ and suppose poles of $\phi_i$ have order $i$, generic residues.

1) Then $F_{wu}$ gives map of moduli space.

\[
\mathcal{F}_{wu} : \tilde{\mathcal{M}}(\Sigma_u, \mathbb{C}G(1)) \to \mathcal{M}(C, \mathbb{C}G(N))
\]

\[
\xymatrix{ \mathcal{U}_W \ar[r]^-{F_{wu}} & \tilde{\mathcal{M}}(\Sigma_u, \mathbb{C}G(1)) \ar[d]^-{\chi^W} \ar[r]^-{X_W} & \mathbb{C}^x } 
\]

2) $F_{wu}$ is local symplectomorphism $\Rightarrow$ locally invertible:

\[
\mathcal{M}(C, \mathbb{C}G(N)) 
\]

\[
\left[ \varphi \in H_1(\Sigma, \mathbb{Z}) \right] 
\]

\[
X^W \cdot X^W = \pm X^W
\]

\[
X^W(\nabla^0) = \text{Hol}_x \nabla^0 \in \mathbb{C}^x
\]

The $X^W$ give local coordinates on $\mathcal{M}(C, \mathbb{C}G(N))$

- For $N=2$, $\phi_2$ generic, $X^W$ are Fock-Goncharov cards.
- For $N=3$, $\phi_3 < \phi_2$, $\phi_2$ generic, $X^W$ are higher-rank Fock-Goncharov cards.

3) $F_{wu}$ jumps when $W$ jumps. Simpler example: $(N=2)$

In this example one computes $F_{wu} = F_{wu} \circ K_y$ where $K_y : \tilde{\mathcal{M}}(\Sigma_u, \mathbb{C}G(1)) \to \tilde{\mathcal{M}}(\Sigma_u, \mathbb{C}G(1))$

\[
K_y \cdot X^W = X^W (1 + X_y)
\]

**Slogan**: a saddle connection of charge $\gamma$ induces the transformation $K_y$

Similarly a 3-shingled web of charge $\gamma$ 

---

\[
\text{sorry}
\]

\[
\text{ring domain}
\]
More generally, at any $P$ where $W(u, \theta)$ jumps, $\frac{d}{du}$ changes by a $z$-form $T_{\theta, u} : \tilde{\mathcal{M}}(\mathbb{R} \tilde{C}(U)) \to \tilde{\mathcal{M}}(\mathbb{R} \tilde{C}(U'))$.

Def. $u$ is not on a wall of marginal stability if $\arg Z_y(u) = \arg Z_y(u') \iff \theta_k = \theta_k'$.

Fact. If $u$ is not on a wall of marginal stability, then

$\exists \Omega(\gamma, u) \in \mathbb{Z}$ such that $T_{\theta, u} = \prod_{\gamma : \arg Z_y = \theta_k} K_{\gamma}$ with $K_{\gamma}$ defined as above.

There's an algorithm computing the $\Omega(\gamma, u)$ from "degenerate spectral network" $W(u, \theta)$.

Call the $\Omega(\gamma, u)$ "DT-like invariants." (For $N=2$ and $\phi_2$ meromorphic w/poles order $\geq 2$ they are DT invariants.)

Remark

In simple cases $\Omega(\gamma, u)$ "count" finite webs with charge $\gamma$.

eg for $N=2$, $\Omega(\gamma, u) = \left( \# \text{saddle connections of } \phi_2 \text{ with charge } \gamma \right) - 2 \left( \# \text{rig domains of } \phi_2 \text{ with charge } \gamma \right)$

But in general there can be lots of overlapping finite webs, and then it can be hard to disentangle their individual contributions to $\Omega(\gamma, u)$.

A "bad" example:

\[
\begin{bmatrix}
 & \mu_2 & \nu_3 \\
\end{bmatrix}
\]

If 3-herd appears at some $\theta_k$ the corresponding DT invariants are:

\[
(\Omega(n\gamma))_n^\infty = 3, -6, 18, -84, 465, -2808, \ldots
\]

where $\gamma = \text{lift of the finite web} \rightarrow$.
Let's take $N=2$, $C=\mathbb{CP}^1$, $\phi^2=(z^3-2z+u)\,dz^2$. Finely many saddle connections for any $u$.

\[
\begin{align*}
\text{at } u=u_1 &\approx 0, \quad \phi^2=(z^3-z)\,dz^2 \\
\Omega(\pm\delta_1, u_1) &= 1 \\
\Omega(\pm\delta_2, u_1) &= 1 \\
\text{all other } \Omega(\delta, u_1) &= 0
\end{align*}
\]

\[
\begin{align*}
\text{at } u=u_2 &\approx \frac{1}{2}, \quad \phi^2=(z^3-z+\frac{1}{2})\,dz^2 \\
\Omega(\pm\delta_1, u_2) &= 1 \\
\Omega(\pm\delta_2, u_2) &= 1 \\
\Omega(\pm(\delta_1+\delta_2), u_2) &= 1 \\
\text{all other } \Omega(\delta, u_2) &= 0
\end{align*}
\]

Now plot $(\Sigma, u)$ space with a wall at every place that $\Omega(\delta, u)$ jumps.

Each wall is carrying an automorphism of $\widetilde{\mathcal{M}}(\Sigma, \mathcal{G}(1))$. But $\widetilde{\mathcal{M}}(\mathcal{G}(n))$ depends only on $(\Sigma, u)$ so considering the two paths shown gives the identity:

$$K_{\psi_2} \circ K_{\psi_1} = K_{\psi_1} \circ K_{\psi_1} \circ K_{\psi_2}$$

The same property should hold much more generally for our $\Omega(\delta, u)$, [even in $N>2$ cases]:

$$K_{\psi_1} \circ \Omega(\delta, u_1) = \Omega(\delta, u_1) \circ K_{\psi_1}$$

This is enough to determine the $\Omega(\delta, u_2)$ from the $\Omega(\delta, u_1)$ — this is Korteweg–Saito–Sakurai wall-crossing formula.