M340L First Midterm Exam Solutions, February 18, 2010

1.Let $A = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 2 & 1 & 7 & 6 \\ 3 & 3 & 12 & 6 \end{pmatrix}$.

a) Compute $A_{rref}$.

$$A_{rref} = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) Find all solutions to $A\mathbf{x} = 0$.

The equations become $z_1 = -3x_3 - 4x_4$, $x_2 = -x_3 + 2x_4$, with $x_3 = x_3$ and $x_4 = x_4$. In other words $\mathbf{x} = x_3 \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 2 \\ 0 \\ 1 \end{pmatrix}$, where $x_3$ and $x_4$ are arbitrary.

c) Find all solutions to $A\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

This is similar, only with an extra term on the right-hand side. Row-reducing $[A|\mathbf{b}]$ gives $\begin{pmatrix} 1 & 0 & 3 & 4 & | & -1 \\ 0 & 1 & 1 & -2 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$. This means $z_1 = -3x_3 - 4x_4 - 1$, $x_2 = -x_3 + 2x_4 + 1$, with $x_3 = x_3$ and $x_4 = x_4$. In other words $\mathbf{x} = x_3 \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, where $x_3$ and $x_4$ are arbitrary.

d) Find all solutions to $A\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Row-reducing $[A|\mathbf{b}]$ gives $\begin{pmatrix} 1 & 0 & 3 & 4 & | & -1 \\ 0 & 1 & 1 & -2 & | & 1 \\ 0 & 0 & 0 & 0 & | & 1 \end{pmatrix}$, so there are no solutions.

2. Consider the vectors $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{a}_3 = \begin{pmatrix} 5 \\ 7 \\ 12 \end{pmatrix}$, $\mathbf{a}_4 = \begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix}$,

$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

a) Are the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ linearly independent? If not, write the zero vector as a nontrivial linear combination of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$.

This is really the same question as 1b. The vectors are linearly dependent since there are nontrivial solutions to $A\mathbf{x} = 0$. For instance, $-3\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 = 0$. Also, $-4\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_4 = 0$.

b) Is $\mathbf{u}$ in the span of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$? If so, write $\mathbf{u}$ as a linear combination of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$.

This is the same thing as 1c. $\mathbf{u}$ is in the span, since there are solutions to $A\mathbf{x} = \mathbf{u}$. In particular, $-\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{u}$.

c) Is $\mathbf{v}$ in the span of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$? If so, write $\mathbf{v}$ as a linear combination of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$.

This is the same thing as 1d. $\mathbf{v}$ is not in the span, as there are no solutions to $A\mathbf{x} = \mathbf{v}$.

3. (a) Let $T_1 : R^2 \to R^2$ be a linear transformation that first rotates vectors 90 degrees counterclockwise and then reflects them across the vertical axis. Find the (standard) matrix of $T_1$. Is $T_1$ 1-1? If $T_1$ onto?

Since $T_1(\mathbf{e}_1) = \mathbf{e}_2$ and $T_1(\mathbf{e}_2) = \mathbf{e}_1$, the matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This has rank 2, so $T_1$ is both 1-1 and onto.

(b) Let $T_2 : R^2 \to R^3$ be given by the formula $T_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 - x_2 \\ 0 \\ 9x_1 - 3x_2 \end{pmatrix}$. Find the (standard) matrix of $T_2$. Is $T_2$ 1-1? If $T_2$ onto?

The matrix is $\begin{pmatrix} T_2(\mathbf{e}_1) & T_2(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 0 & 0 \\ 9 & -3 \end{pmatrix}$. This row-reduces to something with only one pivot, so it is neither 1-1 nor onto. (If it had two pivots, it would be 1-1 but not onto.)

4. (a) Does $\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$ have an inverse? If so, find it.

No. The determinant is zero.

(b) Does $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$ have an inverse? If so, find it.

Yes. By row-reducing $[A|I]$ we get that $A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}$.

5. True/false. Just mark each statement with a T (or TRUE) or an F (or FALSE). You do not need to justify your answers, and partial credit will not be given.

a) If $A$ is a left-inverse, then the columns of $A$ are linearly independent.

TRUE. If $A\mathbf{x} = 0$, then $\mathbf{x} = LA\mathbf{x} = L0 = 0$.

b) If a $3 \times 5$ matrix $A$ has rank 3, then the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.

TRUE. There is a pivot in each row.

c) If $A\mathbf{x} = 0$ has infinitely many solutions, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

FALSE. There may not be any solutions. (See problem 1d).

d) If $T(\mathbf{x}) = A\mathbf{x}$, then the third column of $A$ (assuming $A$ has at least three columns, of course) is $T(\mathbf{e}_3)$.

TRUE. $T(\mathbf{e}_3) = A\mathbf{e}_3$ is the third column of $A$.

e) If $A$ is a $3 \times 5$ matrix, then the columns of $A$ are linearly dependent.

TRUE. There's no way for $A$ to have 5 pivots if it only has 3 rows.

f) If $A$ and $B$ are row-equivalent matrices, then the equations $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{b}$ have the same solutions.

FALSE. When converting $A$ to $B$ by row operations, you typically change the right hand side of the equations.

g) The columns of $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{pmatrix}$ span $R^5$.

FALSE. There are only four vectors, and four vectors can't span $R^5$.

h) If four vectors in $R^4$ are linearly independent, then they span $R^4$.

TRUE. For collections of $n$ vectors in $R^n$, linear independence is equivalent to spanning.

i) If a square matrix has a right-inverse, then the columns are linearly independent.

TRUE. If a square matrix has a right-inverse, then it is invertible and has a left-inverse, too.

j) If $A$ and $B$ are $2 \times 2$ matrices, then $AB = BA$.

FALSE. Matrices typically do not commute.