M 340L Fall 2010 (55430), Midterm 2

Answer the questions on the scratch paper provided. Box your answers. Correct answers will always get full credit even if no work is shown; if you want partial credit, show your work. All scratch paper will be collected at the end.

Name: ______________________________

Question	Points	Score
1	36	
2	16	
3	30	
4	34	
5	30	
6	40	
7	60	
Total:	246	

Problem 1 (36 points).

Consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 5 & 2 & 0 \end{bmatrix}.$$

(a) (10 points) Find a basis for Row A.

Solution: Row reduction of A gives the row-equivalent matrix

$$\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot rows are the first three rows. A basis for Row A thus consists of these three rows. Since we always write vectors in $\mathbb{R}^n$ as columns, this gives the basis as:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(It would actually be equally good — in some sense better — to write them as rows instead of columns. Also, this is not the only possible basis: we need only put A in row echelon form, not necessarily *reduced* row echelon form.)

(b) (10 points) Find a basis for Nul A.

Solution: Using the reduced row echelon form from the previous part, the equation $A\mathbf{x} = \mathbf{0}$ is equivalent to

$$\begin{aligned} x_1 &= -8x_4, \\ x_2 &= 2x_4, \\ x_3 &= -x_4, \end{aligned}$$

with x_4 free. We thus have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -8 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

so a basis for Nul A is

$$\left\{ \begin{bmatrix} -8 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(c) (10 points) Find a basis for $\operatorname{Col} A$.

> **Solution:** A basis for $\operatorname{Col} A$ is provided by the pivot columns of the *original matrix* A:
> $$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

(d) (6 points) What is the dimension of the range of the linear transformation T from $\mathbb{R}^4$ to $\mathbb{R}^4$ given by $T(\mathbf{x}) = A\mathbf{x}$?

> **Solution:** The range of T is the same as the row space of A: $\operatorname{Ran} T = \operatorname{Row} A$. So $\dim \operatorname{Ran} T = \dim \operatorname{Row} A$. But we just saw in part a that $\operatorname{Row} A$ has a basis of 3 vectors, so $\dim \operatorname{Ran} T = \dim \operatorname{Row} A = 3$. (Of course, this is also equal to $\dim \operatorname{Col} A$: these two dimensions are always equal, and also known as the rank of A.)

Problem 2 (16 points).

Consider the matrix
$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{bmatrix}.$$

(a) (10 points) Calculate $\det A$.

> **Solution:** The simplest way is by cofactor expansion on the last column, since that column has three zeroes. That gives
> $$\det A = -2 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 1 & -1 & 1 \end{vmatrix}.$$
> To evaluate this smaller determinant, you can either use the cofactor expansion again, or else use the formula for a 3×3 determinant, which gives
> $$(1\cdot 0\cdot 1 + 2\cdot 2\cdot 1 + 3\cdot 2\cdot -1) - (1\cdot 2\cdot -1 + 2\cdot 2\cdot 1 + 3\cdot 0\cdot 1) = (4-6) - (-2+4) = -4$$
> so $\det A = -2 \cdot -4 = 8$.

(b) (6 points) Are the rows of A linearly independent? Explain in a few words how you know.

Solution: Yes, they are linearly independent. The reason is that $\det A \neq 0$, which is equivalent to A being invertible, which is equivalent to the rows of A being linearly independent.

Problem 3 (30 points).

Let $\mathbb{P}_2$ be the vector space of polynomials with real coefficients and degree at most 2. Consider two bases for $\mathbb{P}_2$:

$$\mathcal{B} = \{1, t, t^2 - 1\}, \qquad \mathcal{C} = \{1, t, t^2 + 1\}.$$

(a) (8 points) What is the polynomial p with $\mathcal{B}$-coordinate vector $[p]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$?

Solution: By definition, p is a linear combination of the basis vectors in $\mathcal{B}$, with coefficients given by the entries of $[p]_{\mathcal{B}}$. So here

$$p = 2 \cdot (1) + 1 \cdot (t) + -1 \cdot (t^2 - 1) = -t^2 + t + 3.$$

(b) (12 points) If q is the polynomial $t^2 - 1$, what is the $\mathcal{C}$-coordinate vector $[q]_{\mathcal{C}}$?

Solution: This question asks: $t^2 - 1$ is a linear combination of the basis vectors in $\mathcal{C}$, with what coefficients? In other words what are x_1, x_2, x_3 such that

$$t^2 - 1 = x_1 \cdot (1) + x_2 \cdot (t) + x_3 \cdot (t^2 + 1)\,?$$

One can see "by inspection" that this equation is solved by $x_1 = -2$, $x_2 = 0$, $x_3 = 1$. (Or, one can write it out as a system of linear equations and solve them as usual.) This means that

$$[q]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

(c) (10 points) Find the change-of-basis matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$.

Solution: Let b_1, b_2, b_3 denote the three elements of $\mathcal{B}$, i.e. $b_1 = 1$, $b_2 = t$, $b_3 = t^2 - 1$. Let c_1, c_2, c_3 denote the three elements of $\mathcal{C}$, i.e. $c_1 = 1$, $c_2 = t$, $c_3 = t^2 + 1$. In general the change-of-basis matrix is

$$P_{\mathcal{C} \to \mathcal{B}} = \begin{bmatrix} [b_1]_{\mathcal{C}} & [b_2]_{\mathcal{C}} & [b_3]_{\mathcal{C}} \end{bmatrix}.$$

In this case $b_1 = c_1$, so $[b_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Also $b_2 = c_2$, so $[b_2]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Finally, in part (b) we found $[b_3]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Putting these together gives

$$P_{\mathcal{C}\to\mathcal{B}} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 4 (34 points).

The matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 0 & 3 \\ 0 & 1 & 4 & 2 & 1 \\ 1 & 3 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 1 & 2 & 0 & 3 \end{bmatrix}$$

has

$$\det A = 1.$$

Consider the matrix

$$B = \begin{bmatrix} 1 & 2 & 2 & 0 & 3 \\ 0 & 1 & 4 & 2 & 1 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 1 & 2 & 0 & 3 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix}$$

which is related to A by exchanging the last two rows.

(a) (8 points) What is $\det B$?

Solution: Since A and B are related by exchanging two rows,

$$\det B = -\det A = -1.$$

(b) (8 points) What is the rank of A?

Solution: Since $\det A \neq 0$, A is invertible, so A has full rank — in this case 5 since A is a 5×5 matrix.

(c) (8 points) What is the rank of $A - B$?

Solution: One way to do this is to row-reduce the matrix

$$A - B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & -2 \\ 1 & 0 & 1 & -1 & 2 \end{bmatrix}$$

and see that it has 1 pivot, so $\operatorname{rank} A - B = 1$.

A slightly more conceptual way of thinking about it is as follows. The first three rows of $A - B$ are all zero, while the bottom two rows are $\mathbf{v}$ and $-\mathbf{v}$ for some vector $\mathbf{v}$ in $\mathbb{R}^5$. Hence $\operatorname{Row} A - B = \operatorname{Span}\{\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{v}, -\mathbf{v}\}$. But the three $\mathbf{0}$ are linear combinations of the other vectors (in a trivial way) and also $-\mathbf{v}$ is a linear combination of the set $\{\mathbf{v}\}$. So we can throw all these away to get a basis $\{\mathbf{v}\}$ for $\operatorname{Row} A - B$. So $\operatorname{Row} A - B$ is 1-dimensional, i.e. $\operatorname{rank} A - B = 1$.

Let H be the set of all vectors $\mathbf{x}$ in $\mathbb{R}^5$ such that $A\mathbf{x} = B\mathbf{x}$.

(d) (10 points) Is H a subspace of $\mathbb{R}^5$? If so, what is its dimension? If not, explain in a few words why.

Solution: H is the set of all vectors $\mathbf{x}$ in $\mathbb{R}^5$ obeying $(A - B)\mathbf{x} = \mathbf{0}$, or in other words,

$$H = \operatorname{Nul}(A - B).$$

So H is indeed a subspace of $\mathbb{R}^5$. Since $\operatorname{rank}(A - B) + \dim \operatorname{Nul}(A - B) = 5$, and we know from part (c) that $\operatorname{rank}(A - B) = 1$, we get $\dim \operatorname{Nul}(A - B) = 4$, i.e.

$$\dim H = 4.$$

Problem 5 (30 points).

Consider the vector space

$$W = \mathbb{P}_6 = \{\text{polynomials } p(t) \text{ of degree less than or equal to 6, with real coefficients}\}.$$

Let U be the linear transformation from W to W given by

$$U(p) = p''.$$

(Here p'' denotes the second derivative of p.)

(a) (15 points) Find a basis for $\operatorname{Ker} U$. What is the dimension of $\operatorname{Ker} U$?

Solution: $\operatorname{Ker} U$ is the set of all $p \in \mathbb{P}_6$ such that $U(p) = \mathbf{0}$, i.e. p'' is the zero polynomial. Any p for which p'' is the zero polynomial can only have constant and linear terms, but no higher: i.e.

$$p = a + bt = a \cdot 1 + b \cdot t.$$

In other words, any $p \in \operatorname{Ker} U$ is a linear combination of the polynomials 1 and t. So $\operatorname{Ker} U = \operatorname{Span}\{1, t\}$. Also, the set $\{1, t\}$ is linearly independent. So $\{1, t\}$ is a basis for $\operatorname{Ker} U$. Since $\operatorname{Ker} U$ has a basis with 2 elements, $\dim \operatorname{Ker} U = 2$.

(b) (15 points) Find a basis for $\operatorname{Ran} U$. What is the dimension of $\operatorname{Ran} U$?

Solution: $\operatorname{Ran} U$ is the set of all $q \in \mathbb{P}_6$ such that $q = U(p)$ for some p. If we let p be any element in $\mathbb{P}_6$,

$$p = a + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6,$$

then

$$q = p'' = (2c) \cdot 1 + (6d) \cdot t + (12e) \cdot t^2 + (20f) \cdot t^3 + (30g) \cdot t^4.$$

In other words, any $q \in \operatorname{Ran} U$ is a linear combination of the polynomials $1, t, t^2, t^3, t^4$. So $\operatorname{Ran} U = \operatorname{Span}\{1, t, t^2, t^3, t^4\}$. Also, the set $\{1, t, t^2, t^3, t^4\}$ is linearly independent. So $\{1, t, t^2, t^3, t^4\}$ is a basis for $\operatorname{Ran} U$. Since $\operatorname{Ran} U$ has a basis with 5 elements, $\dim \operatorname{Ran} U = 5$.

A useful check on the answer is $\dim \operatorname{Ker} U + \dim \operatorname{Ran} U = \dim \mathbb{P}_6 = 7$.

Problem 6 (40 points).

Consider the matrix

$$A = \begin{bmatrix} 5 & -3 & 0 \\ 6 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) (15 points) Find all eigenvalues of A.

Solution: First write the matrix

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -3 & 0 \\ 6 & -4-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}.$$

Then by cofactor expansion using the last column,

$$\begin{aligned}\det(A-\lambda I) = (\lambda-2)\begin{vmatrix} 5-\lambda & -3 \\ 6 & -4-\lambda \end{vmatrix} &= (\lambda-2)((\lambda-5)(\lambda+4)-(-3)(6)) \\ &= (\lambda-2)(\lambda^2-\lambda-2) \\ &= (\lambda-2)(\lambda-2)(\lambda+1).\end{aligned}$$

So the eigenvalues are $\lambda = 2$ (with algebraic multiplicity 2) and $\lambda = -1$ (with algebraic multiplicity 1).

(b) (10 points) Find the eigenspace of A corresponding to the eigenvalue 2.

Solution: This eigenspace is $\mathrm{Nul}(A-2I)$, so we have to solve $(A-2I)\mathbf{x} = \mathbf{0}$. First write the matrix

$$A - 2I = \begin{bmatrix} 3 & -3 & 0 \\ 6 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Row reduction then gives

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding equations are

$$x_1 = x_2$$

with x_2 and x_3 free, so

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So a basis for the eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c) (5 points) Find all other eigenspaces of A.

Solution: The only other eigenvalue is $\lambda = -1$. This eigenspace is $\mathrm{Nul}(A+I)$, so we have to solve $(A+I)\mathbf{x} = \mathbf{0}$. First write the matrix

$$A + I = \begin{bmatrix} 6 & -3 & 0 \\ 6 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Row reduction then gives

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding equations are

$$\begin{aligned} x_1 &= \frac{1}{2} x_2 \\ x_3 &= 0 \end{aligned}$$

with x_2 free, so

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}.$$

So a basis for the eigenspace is

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(d) (10 points) Is there a matrix P and a diagonal matrix D such that $P^{-1}AP = D$? If so, write one such P and the corresponding D.

Solution: We've just seen that A admits 3 linearly independent eigenvectors, namely

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix},$$

with corresponding eigenvalues

$$\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = -1.$$

So A is indeed diagonalizable. The matrix P is built from the eigenvectors:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The diagonal matrix D is built from the eigenvalues, taken in the same order:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Problem 7 (60 points).

True or False. If a statement is sometimes true and sometimes false, write "false". You do not have to justify your answers.

(a) (6 points) If A is an $n \times n$ matrix, then $\operatorname{Row} A = \operatorname{Col} A$.

Solution: FALSE. The two spaces have the same dimension, but they don't have to be the same space, even if A is $n \times n$. For an explicit counterexample, contemplate the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) (6 points) If V is a vector space of dimension n, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subset of V with $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = V$, then $p \geq n$.

Solution: TRUE. A set of p vectors spans a space of dimension $\leq p$.

(c) (6 points) If V is a 4-dimensional vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ is a subset of V such that $\operatorname{Span} S = V$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis of V.

Solution: FALSE. It's true that we can throw away *some* vector from S to get a basis for V, but $\mathbf{v}_5$ might not be the right one to throw away. The rule is that we can only throw away a vector which is a linear combination of the other vectors. Even though S is linearly dependent, it might happen that $\mathbf{v}_5$ is not a linear combination of the others.

(d) (6 points) Any set of 4 linearly independent vectors in $\mathbb{R}^4$ forms a basis of $\mathbb{R}^4$.

Solution: TRUE. We emphasized in lecture that for a set of n vectors in $\mathbb{R}^n$, one doesn't have to check both linear independence and spanning separately — either one on its own is enough to imply that the set is a basis.

(e) (6 points) A plane P in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ if and only if P contains the zero vector.

Solution: TRUE. A plane through the origin satisfies all of the subspace axioms, but a plane not through the origin does not: indeed one of our axioms was that a subspace always contains the zero vector.

(f) (6 points) If A is a 10×20 matrix with rank 8, then the space of solutions to the equation $A\mathbf{x} = \mathbf{0}$ is 12-dimensional.

> **Solution:** TRUE. The space of solutions to $A\mathbf{x} = \mathbf{0}$ is $\operatorname{Nul} A$. For a 10×20 matrix, we have $\dim \operatorname{Nul} A + \operatorname{rank} A = 20$. So here $\dim \operatorname{Nul} A = 12$.

(g) (6 points) Every $n \times n$ matrix has n distinct eigenvalues.

> **Solution:** FALSE. It's true that the characteristic polynomial has degree n, so there would be n eigenvalues if we count algebraic multiplicity, *and* include complex eigenvalues. But there is no reason why the eigenvalues would always be distinct. (Problem 6 is a counterexample.)

(h) (6 points) If $\mathbf{v}$ is an eigenvector of A and $\mathbf{v}$ is an eigenvector of B, then $\mathbf{v}$ is an eigenvector of AB.

> **Solution:** TRUE. If $A\mathbf{v} = \lambda\mathbf{v}$ and $B\mathbf{v} = \mu\mathbf{v}$, then
>
> $$AB\mathbf{v} = A(B\mathbf{v}) = A(\mu\mathbf{v}) = \mu(A\mathbf{v}) = \mu(\lambda\mathbf{v}) = (\lambda\mu)\mathbf{v}$$
>
> so $\mathbf{v}$ is an eigenvalue of AB, with eigenvalue $\lambda\mu$.

(i) (6 points) If a matrix A has 0 as an eigenvalue then A is not diagonalizable.

> **Solution:** FALSE. 0 is as good as any other eigenvalue from the point of view of diagonalization. It is true that if A has 0 as an eigenvalue then A is not *invertible*, but that's another issue altogether.

(j) (6 points) If A and B are matrices which can be multiplied, then $\operatorname{rank} AB \leq \operatorname{rank} A$.

> **Solution:** TRUE. This last one was intended to be a bit difficult; don't worry too much if you understand all the others but not this one. One way of understanding it is to think about the rank as the dimension of the column space. $\operatorname{Col} A$ is the set of all vectors of the form $A(\mathbf{x})$. $\operatorname{Col} AB$ is the set of all vectors of the form $AB(\mathbf{y})$, or $A(B\mathbf{y})$. But any vector of the form $A(B\mathbf{y})$ is certainly of the form $A(\mathbf{x})$: just take $\mathbf{x} = B\mathbf{y}$! So $\operatorname{Col} AB$ is actually a *subspace* of $\operatorname{Col} A$. That means $\dim \operatorname{Col} AB \leq \dim \operatorname{Col} A$, i.e. $\operatorname{rank} AB \leq \operatorname{rank} A$.