Problem 1.

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

(a) Find the solution set of $A\mathbf{x} = \mathbf{0}$. Give it in parametric form (unless it is empty).

(b) Is the solution set of $A\mathbf{x} = \mathbf{0}$ a subspace of $\mathbb{R}^3$? If so, what is its dimension?

(c) Consider the vector

$$\mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

Find the solution set of $A\mathbf{x} = \mathbf{b}$. Give it in parametric form (unless it is empty).

(d) Is the solution set of $A\mathbf{x} = \mathbf{b}$ a subspace of $\mathbb{R}^3$? If so, what is its dimension?

Problem 2.

Suppose V is a vector space with $\dim V = 3$, W is a vector space with $\dim W = 7$, and T is a linear transformation from V to W.

(a) Is $\operatorname{Ran} T$ a subspace of W? If so, what are all the possibilities for its dimension?

(b) Is $\operatorname{Nul} T$ a subspace of V? If so, what are all the possibilities for its dimension?

Problem 3.

Consider the three vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 9 \\ -15 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 7 \\ -10 \end{bmatrix}.$$

(a) Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for $\mathbb{R}^3$? Why or why not?

Let $S = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) Find a basis for S. What is the dimension of S?

(c) Find an *orthogonal* basis for S.

Consider the vector

$$\mathbf{y} = \begin{bmatrix} -10 \\ 10 \\ 4 \end{bmatrix}.$$

(d) Find the orthogonal projection of $\mathbf{y}$ onto S.

Problem 4.

Consider the matrix

$$A = \begin{bmatrix} -5 & -6 & 0 \\ \frac{9}{2} & 7 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

(a) Find all eigenvalues of A.

(b) For each eigenvalue, describe the corresponding eigenspace.

(c) Is there a matrix P such that $PAP^{-1} = D$ with D diagonal? If so, write P and D.

Problem 5.

Consider the *symmetric* matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Find an *orthonormal* basis of $\mathbb{R}^2$, consisting of eigenvectors of A.

Consider the quadratic form

$$Q(x, y) = x^2 + 6xy + y^2.$$

(c) Is the point $(x, y) = (0, 0)$ a minimum, maximum or saddle point?

(d) Find any values of (x, y) on the unit circle where $Q(x, y)$ is minimized and maximized.

Problem 6.

Consider the dynamical system

$$A\mathbf{x}_n = \mathbf{x}_{n+1}$$

where A is a 2×2 matrix whose (complex) eigenvalues are $\lambda = 2 + i$ and $\lambda = 2 - i$.

(a) Is the origin attracting, repelling, saddle point or none of the above?

(b) Are there any possible starting points $\mathbf{x}_0$ for which the $\mathbf{x}_n$ all lie on a line through the origin?

Now consider instead

$$A^2\mathbf{x}_n = \mathbf{x}_{n+1}.$$

(c) Now is the origin attracting, repelling, saddle point or none of the above?

Problem 7.

Let $\mathcal{F}$ be the vector space consisting of all real-valued functions $f(t)$ on the real line. Let V be the subspace of $\mathcal{F}$ defined by $V = \text{Span}\{\sin t, \cos t\}$. Let $T : V \to \mathbb{R}^3$ be the linear transformation

$$T(f) = \begin{bmatrix} f(0) \\ f(\pi) \\ f(2\pi) \end{bmatrix}.$$

(a) Find a basis for $\text{Ran}\, T$. What is the dimension of $\text{Ran}\, T$?

(b) What is the dimension of $\text{Ker}\, T$?

Problem 8.

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) What is $\det A$?

(b) What are the eigenvalues of A?

(c) What is the dimension of the 1-eigenspace of A?

(d) Is A diagonalizable?

Consider the matrix

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

(e) Is B diagonalizable? (Hint: no calculations are needed.)

Problem 9.

True or False. If a statement is sometimes true and sometimes false, write "false". You do not have to justify your answers.

(a) If a matrix has orthogonal columns, then it is an orthogonal matrix.

(b) If 3 vectors form an orthogonal set then they are linearly independent.

(c) If $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and $\mathbf{v}_1 \cdot \mathbf{v}_3 = 0$ then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set.

(d) Every subspace of $\mathbb{R}^n$ has an orthonormal basis.

(e) If V is a vector space with a subspace W, U is another vector space, and $T : V \to U$ is a linear transformation, then $\{\mathbf{u} : \mathbf{u} = T(\mathbf{w}) \text{ for some } \mathbf{w} \in W\}$ is a subspace of U.

(f) If the equation $A\mathbf{x} = \mathbf{b}$ admits a solution $\mathbf{x}$, and A is an $n \times n$ invertible matrix, then the solution is unique.

(g) If V and W are subspaces of $\mathbb{R}^4$ then their intersection is also a subspace of $\mathbb{R}^4$.

(h) If $\det A = 1$ then the eigenvalues of A are all 1.

(i) If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal then $\|\mathbf{v}\| + \|\mathbf{w}\| = \|\mathbf{v} + \mathbf{w}\|$.