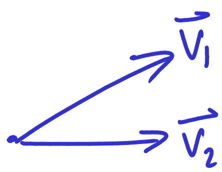
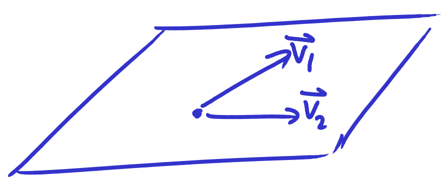
Exam Review

Span: Suppose have a set of vectors $\vec{v}_1, \dots, \vec{v}_k$ in $\mathbb{R}^n$.
Then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is the set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_k$.

In other words: $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is the set of all vectors $\vec{u}$ of the form $\vec{u} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k$.





$\text{Span}\{\vec{v}_1, \vec{v}_2\}$

Given a $m \times n$ matrix A we may ask:

$$m \text{ rows}\left\{ \overbrace{\begin{bmatrix} \quad & & \quad \\ & & \\ & & \end{bmatrix}}^{n \text{ cols}} \right.$$

Do the columns of A span $\mathbb{R}^m$?

i.e. if $A = \begin{bmatrix} \vec{a}_1 \cdots \vec{a}_n \end{bmatrix}$, is $\text{Span}\{\vec{a}_1, \dots, \vec{a}_k\} = \mathbb{R}^m$?

i.e. is every vector in $\mathbb{R}^m$ of the form $x_1\vec{a}_1 + \cdots + x_k\vec{a}_k$?

i.e. " " " " " " " " $A\vec{x}$ (where $\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$)?

i.e. can we always solve $A\vec{x} = \vec{b}$ for any $\vec{b}$?

i.e. does A have a pivot in every row?

T/F: If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is lin. dep.

and T is a linear transformation

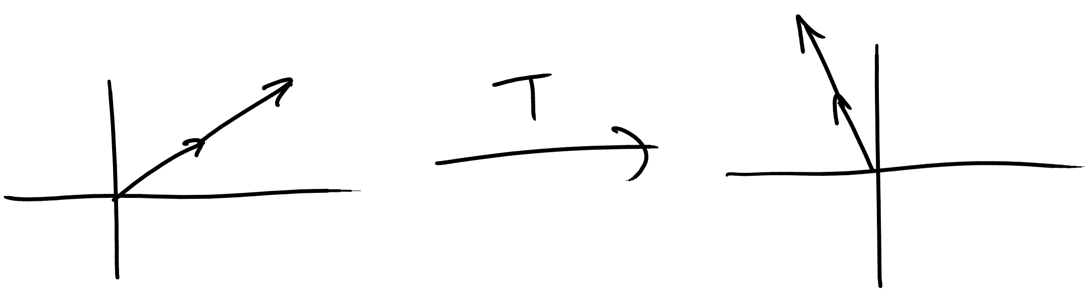
then $\{T(\vec{v}_1), \dots, T(\vec{v}_k)\}$ is lin. dep.

Lin dep $\Leftrightarrow$ $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{0}$ for some $x_1, \dots, x_k$ not all zero.

$$\Rightarrow T(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k) = T(\vec{0})$$

$$\Rightarrow x_1 T(\vec{v}_1) + x_2 T(\vec{v}_2) + \cdots + x_k T(\vec{v}_k) = \vec{0}$$

So $\{T(\vec{v}_1), \dots, T(\vec{v}_k)\}$ is lin dep. [TRUE]



What is a linear transformation?

It's a map $T: \mathbb{R}^n \to \mathbb{R}^m$

such that I) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$

II) $T(c\vec{x}) = cT(\vec{x})$

Ex $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4x_1+2x_2 \\ 3x_3 \\ 2x_1+x_3 \end{bmatrix}$ is linear

$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_1x_2 \\ 3x_2 \end{bmatrix}$ is not linear

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If $T(\vec{x}) = A\vec{x}$ is not 1-1
then $A\vec{x} = \vec{0}$ has nontrivial solution.

TRUE. Why? If $T(\vec{x}) = A\vec{x}$ is not 1-1
then there exist $\vec{u}, \vec{v}$ with $\vec{u} \neq \vec{v}$
but $T(\vec{u}) = T(\vec{v})$
ie $A\vec{u} = A\vec{v}$

That implies $A(\vec{u}-\vec{v}) = \vec{0}$
So $\vec{x} = \vec{u}-\vec{v}$ is a nontriv. solution of $A\vec{x} = \vec{0}$

"$T(\vec{x}) = A\vec{x}$ is 1-1" is equivalent to
"$A\vec{x} = \vec{0}$ does not have a nontrivial solution" which is also equivalent to
"A has a pivot in every column"

"$T(\vec{x}) = A\vec{x}$ has range equal to $\mathbb{R}^m$" is equivalent to $T: \mathbb{R}^n \to \mathbb{R}^m$
"The columns of A span $\mathbb{R}^m$" which is also equiv. to
"A has a pivot in every row"

NB: "The cols of A span $\mathbb{R}^m$" does <u>not</u> <u>generally</u> mean that the cols of A are lin indep!

<u>Ex</u> $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

The two <u>are</u> equivalent if A is square.

[And if A is square, they're both equiv. to "A is invertible"].

<u>Homogeneous vs. Inhomogeneous</u>:

A lin. sys. $A\vec{x} = \vec{b}$ is called $\begin{cases} \underline{\text{homogeneous}} \text{ if } \vec{b} = \vec{0} \\ \underline{\text{inhomogeneous}} \text{ if } \vec{b} \neq \vec{0} \end{cases}$

Solution set of $A\vec{x} = \vec{0}$ (homog.)

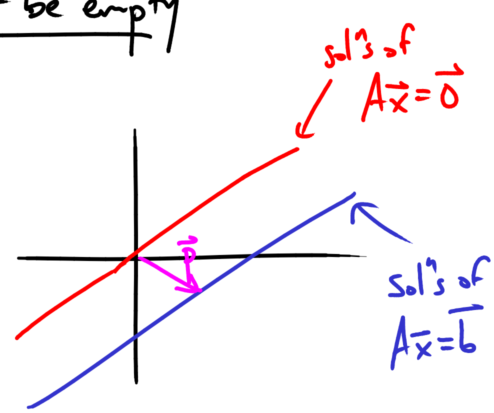
always can be written in parametric form

$$\vec{x} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k \quad \text{for some vectors } \vec{v}_1, \ldots, \vec{v}_k$$

Sol^n set of $A\vec{x} = \vec{b}$ (inhomog.) — <u>might be empty</u>

If not empty, it's of form

$$\vec{x} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k + \vec{p}$$



$T: \mathbb{R}^n \to \mathbb{R}^m$ $\quad$ $\mathbb{R}^n$ is called "domain"
$\mathbb{R}^m$ is called "codomain"

Linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$

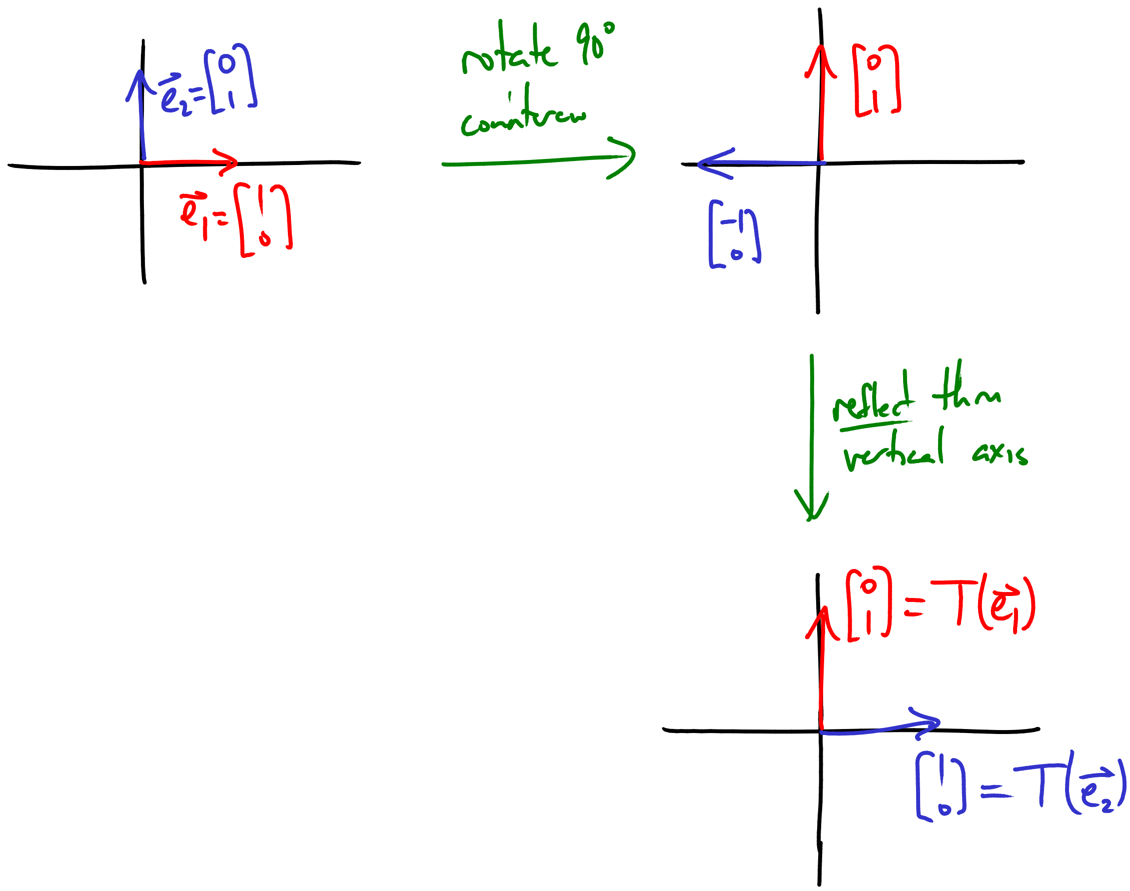
"first rotate 90° counterclockwise then reflect across the vertical axis"

What is the standard matrix of T?

It's a matrix whose columns are $T(\vec{e}_1), T(\vec{e}_2)$

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(\vec{e}_1)$



So the standard matrix is $\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

[i.e. $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}$ for any $\vec{x}$]

Composition of linear transformations:

$$\begin{tikzcd} \vec{x} & U(\vec{x}) & T(U(\vec{x})) \\ \mathbb{R}^p \arrow[r, "U"] \arrow[rr, bend right] & \mathbb{R}^n \arrow[r, "T"] & \mathbb{R}^m \end{tikzcd}$$

If the std. matrix of U is A i.e. $U(\vec{x}) = A\vec{x}$

" " " " T is B i.e. $T(\vec{x}) = B\vec{x}$

What is the matrix C such that

$$T(U(\vec{x})) = C\vec{x} \ ?$$

$$T(U(\vec{x})) = T(A\vec{x}) = B(A\vec{x})$$

$$= (BA)\vec{x}$$

i.e. $\underline{\underline{C = BA}}$

T/F: "If the eq. $A\vec{x} = \vec{b}$ is consistent for some $\vec{b}$ then it is consistent for every $\vec{b}$."

Ex $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$A\vec{x} = \vec{0}$ is consistent (or $A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$)

But $A\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not consistent

So: $\boxed{\text{FALSE}}$

$$\left[\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \longrightarrow \begin{array}{l} x_1 = 0 \\ 0 = 1 \ \times \end{array} \right]$$

Given vectors $\vec{a}_1, \dots, \vec{a}_4$

and $\vec{u}$

Q: Is $\vec{u}$ a lin. comb. of $\vec{a}_1, \dots, \vec{a}_4$?

If so, write it as $\vec{u} = x_1\vec{a}_1 + \dots + x_4\vec{a}_4$.

i.e. solve the eq^n $A\vec{x} = \vec{u}$

where $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \end{bmatrix}$

Suppose $B = PAP^{-1}$.

Find A in terms of B and P.

Wrong: $B = PAP^{-1}$

$\Rightarrow BP^{-1} = P^{-1}PAP^{-1}$

ie $BP^{-1} = AP^{-1}$

Right: $B = PAP^{-1}$ multiply both sides by P^{-1} on the left

$P^{-1}B = P^{-1}(PAP^{-1})$

$P^{-1}B = AP^{-1}$ mult. both sides by P on the right

$P^{-1}BP = AP^{-1}P$

$$\boxed{P^{-1}BP = A}$$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ Is $\{\vec{v}\}$ linearly independent?

$\iff$ Does the equation $x\vec{v} = \vec{0}$ have only the trivial solution?

Yes

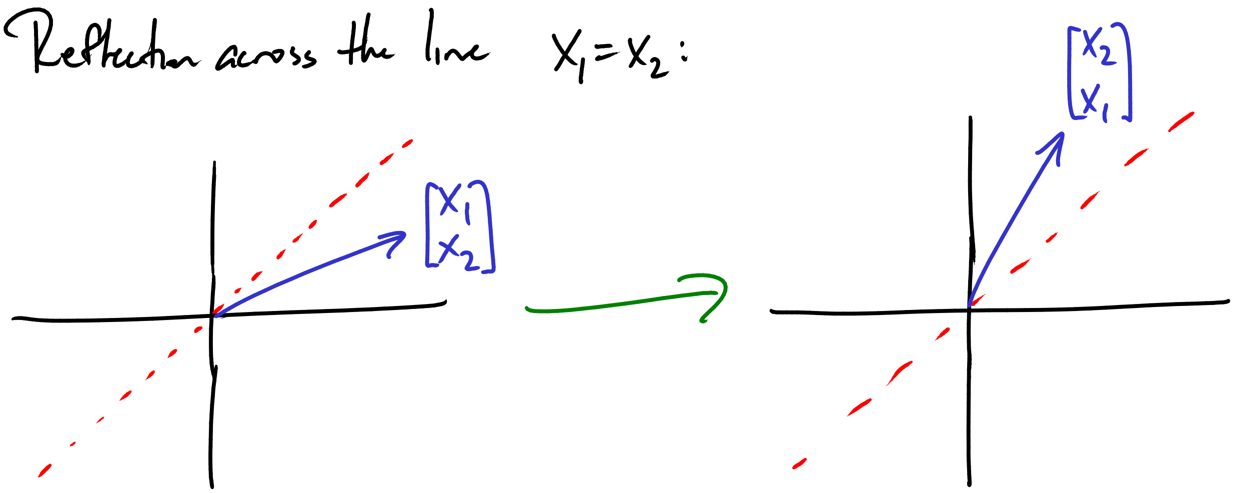
The "trivial solution" of

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$$

is

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ &\vdots \\ x_n &= 0 \end{aligned}$$

Reflection across the line $x_1 = x_2$:



$$T_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_2 \\ 0 \\ 9x_1 - 3x_2 \end{bmatrix}$$

Find the std matrix of T_2.

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - x_2 \\ 0 \\ 9x_1 - 3x_2 \end{bmatrix} = x_1\begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 9 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - x_2 \\ 0 \\ 9x_1 - 3x_2 \end{bmatrix}$$

std matrix

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 9 & -3 \end{bmatrix}$$

"Is T 1-1?" $\Longleftrightarrow$ "Does A have pivots in every column?" <u>No</u>

Row reduce A: $\begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 9 & -3 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$:

"Does T have image = all of $\mathbb{R}^3$?" $\Longleftrightarrow$ "Does A have pivots in every row?" <u>No</u>

Say we have a matrix A. All entries "small".

Then: is $I - A$ invertible?

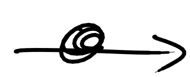
Yes, if A is small enough: $(I-A)^{-1} = I + A + A^2 + A^3 + A^4 + \cdots$

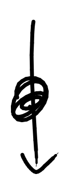
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$(I-A)(I + A + A^2 + A^3 + A^4 + \cdots)$$

$$= I + A - A + A^2 - A^2 + A^3 - A^3 + \cdots$$

$$= I$$

 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

 $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ $\vec{a}_3 = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix}$ $\vec{a}_4 = \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix}$

Are $\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\}$ lin indep?

$\Leftrightarrow$ Does $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + x_4\vec{a}_4 = \vec{0}$ have only the triv. sol^n?

$$A\vec{x} = \vec{0}$$

Row reduce $\left[\begin{array}{cccc|c} 1 & 2 & 5 & 0 & 0 \\ 2 & 1 & 7 & 6 & 0 \\ 3 & 3 & 12 & 6 & 0 \end{array}\right]$ If we get a free var, then have nontriv sol^n.

[But we can only get at most 3 pivots $\Rightarrow$ there must be a free var — don't have to actually do the row reduction!]

So the vectors are lin dep.

$$A\vec{x} = \vec{0}$$

A $AD = I$

Q: Does $A\vec{x} = \vec{b}$ have a solution?

Hint: Look at $A(D\vec{b})$.

$= (AD)\vec{b} = \vec{b}$

Yes: $D\vec{b} = \vec{x}$ is a solution!

So, in ptic, A has a pivot in every row.

So A can't have more rows than columns.

2 vectors $\vec{u}, \vec{v}$: $\vec{u} - \vec{v}$ __is__ a linear comb. of $\vec{u}, \vec{v}$

[Because $\vec{u} - \vec{v} = \vec{u} + (-1) \cdot \vec{v}$]