Lecture 14 | 14 Oct 2010

Last time: some important subspaces (Nul A, Col A, Ker T, Ran T)

Linearly Independent Sets and Bases (Sec 4.3)

Linear independence for subsets of a vector space V is defined just as for subsets of $\mathbb{R}^n$: we say $\{\vec{v}_1, \dots, \vec{v}_p\}$ is lin. indep. if the eq.

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$$

has only the trivial solution $(c_1 = 0, c_2 = 0, \dots, c_p = 0)$.

Ex $\{\sin t, \cos t\}$ is lin indep subset of $\mathcal{F}$.

(Because $c_1 \sin t + c_2 \cos t = 0$ for all t means $c_1 = 0, c_2 = 0$)

$\{\underbrace{\sin^2 t}_{\vec{v}_1}, \underbrace{\cos^2 t}_{\vec{v}_2}, \underbrace{4}_{\vec{v}_3}\}$ is not lin indep subset of $\mathcal{F}$

[Because $\underbrace{1}_{c_1=1} \cdot \sin^2 t + \underbrace{1}_{c_2=1} \cdot \cos^2 t + \underbrace{\left(-\tfrac{1}{4}\right)}_{c_3=-\frac{1}{4}} \cdot 4 = 0$]

Bases

Say V is a vector space.

A basis for V is a set of vectors $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ such that:

- $\mathcal{B}$ is a linearly independent set
- $V = \operatorname{Span} \mathcal{B}$ $\quad (= \operatorname{Span}\{\vec{b}_1, \dots, \vec{b}_p\})$

Ex Say $V = \mathbb{R}^n$ and A is an $n \times n$ matrix.

Then if A is invertible, the columns of A form a basis for $\mathbb{R}^n$. $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$

[Why? Both properties in the def. of basis follow from the Invertible Matrix Theorem:
A has n pivots, so has a pivot in every row $\Rightarrow$ $V = \operatorname{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$
" " " col $\Rightarrow$ $\{\vec{a}_1, \dots, \vec{a}_n\}$ is lin indep]

And conversely, if A is not invertible, then the columns of A do not form a basis for $\mathbb{R}^n$.

Ex Say we pick $A = I$ in the last example.
Its columns $\{\vec{e}_1, \ldots, \vec{e}_n\}$ form a basis for $\mathbb{R}^n$. (e.g. $n=2$ $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$)
("Standard basis" for $\mathbb{R}^n$)

Ex Is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\3\\0\end{bmatrix}\right\}$ a basis for $\mathbb{R}^3$?

Form $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 0 \end{bmatrix}$: is A invertible?

Yes (one way to check: $\det A = -6$)

Ex $V = \mathbb{P}_3 = \{\text{polynomials of degree} \leq 3\}$
$\mathcal{B} = \{1, t, t^2, t^3\}$ is a basis for V.

[Why? • $\mathcal{B}$ lin indep: if $c_1 \cdot 1 + c_2 \cdot t + c_3 \cdot t^2 + c_4 \cdot t^3 = 0$ (for all t)
then $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$

• $\mathcal{B}$ spans V: every poly. of deg ≤ 3 can be written in form $c_1 \cdot 1 + c_2 \cdot t + c_3 \cdot t^2 + c_4 \cdot t^3$]

Fact (Spanning Set Theorem)
Say $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ is a subset of V, $H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$

a) If some vector in S is a linear combination of the others (i.e. if S is linearly dependent), remove that vector to get a set S'. Then $\text{Span}\, S' = H$.

b) Some subset of S is a basis for H.

[Why? a) Say $\vec{v}_p$ is a lin. comb. of the others

$$\vec{v}_p = a_1\vec{v}_1 + \cdots + a_{p-1}\vec{v}_{p-1} \qquad (*)$$

Any $\vec{x} \in H$ is a lin. comb. of $\vec{v}_1, \dots, \vec{v}_p$, i.e.

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_{p-1}\vec{v}_{p-1} + c_p\vec{v}_p$$

Substitute in $(*)$ and get $\vec{x}$ as lin. comb. of $\vec{v}_1, \dots, \vec{v}_{p-1}$.

b) If S is lin dep, throw away a vector which is a lin comb. of the others, to get S'. Span $S' = H$ by part a.

If S' is lin indep, then S' is a basis for H.

If not, repeat...]

Let's apply this to a particular kind of subspace:

$V = \mathbb{R}^m$

A some $m \times n$ matrix

$H = \text{Col } A$ is a subspace of V

How to produce a basis for H?

Fact: The pivot columns of A form a basis for $\text{Col } A$.

[See text for proof]

Ex $V = \mathbb{R}^3$ $\quad A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ 0 & 1 & 4 \end{bmatrix}$. Find a basis for $\text{Col } A$.

Row reduce: $A \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$. So the first 2 cols. are pivot cols.

So $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{Col } A$.

[$\underline{\text{Not}}$ $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}\right\}$ — use columns of the original matrix A!]

In summary: 2 perspectives on a basis –

1) A basis is a spanning set that is as small as possible.
2) A basis is a linearly independent set that is as big as possible.

Coordinate Systems (Sec 4.4)

Say $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis for a vector space V.

Then for any $\vec{x} \in V$ there is a unique set of scalars $c_1, \dots, c_n$

such that $$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \dots + c_n\vec{b}_n.$$

[Why? $\mathcal{B}$ spans V, so $(c_1, \dots, c_n)$ exist.
$\mathcal{B}$ lin independent $\Rightarrow$ $(c_1, \dots, c_n)$ are unique.]

Call $c_1, \dots, c_n$ the $\mathcal{B}$-coordinates of $\vec{x}$

or the coordinates of $\vec{x}$ relative to the basis $\mathcal{B}$,

and define a vector $[\vec{x}]_{\mathcal{B}}$ in $\mathbb{R}^n$ by

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Ex Say $V = \mathbb{R}^2$

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ $\vec{b}_1 = \begin{bmatrix}3\\4\end{bmatrix}$ $\vec{b}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$

Suppose $\vec{x} \in V$ has $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$. What is $\vec{x}$?

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 = -1\begin{bmatrix}3\\4\end{bmatrix} + 2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-1\\-6\end{bmatrix}.$$

<u>Ex</u> Say $V = \mathbb{R}^2$

$$\mathcal{E} = \{\vec{e}_1, \vec{e}_2\} \qquad \vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

Say $\vec{x} \in V$ has $[\vec{x}]_{\mathcal{E}} = \begin{bmatrix}2\\3\end{bmatrix}$. What is $\vec{x}$?

$$\vec{x} = 2\vec{e}_1 + 3\vec{e}_2 = 2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$$

(And more generally $[\vec{x}]_{\mathcal{E}} = \vec{x}$)

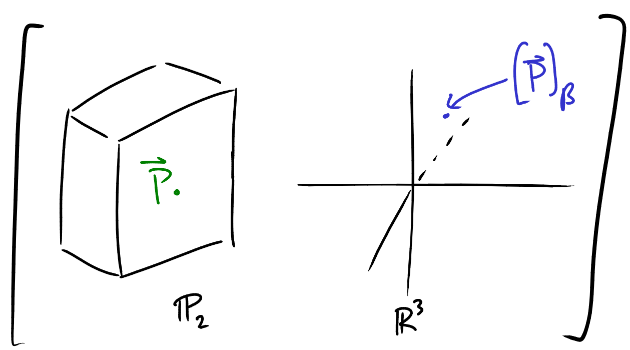
<u>Ex</u> Say $V = \mathbb{P}_2 = \{\text{poly. of degree} \le 2\}$

$$\mathcal{B} = \{1, t, t^2\} \qquad \vec{b}_1 = 1 \quad \vec{b}_2 = t \quad \vec{b}_3 = t^2$$

The vector $\vec{p} = 3 + 5t - t^2$

$$= 3\cdot\vec{b}_1 + 5\cdot\vec{b}_2 - 1\cdot\vec{b}_3$$

So $[\vec{p}]_{\mathcal{B}} = \begin{bmatrix}3\\5\\-1\end{bmatrix}$.



<u>So:</u> a basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ for V gives a recipe for taking vectors $\vec{x}$ in V to vectors $[\vec{x}]_{\mathcal{B}}$ in $\mathbb{R}^n$.

"<u>Coordinate mapping</u>" $\qquad V \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^n$

It's a linear transformation: because

- $[\vec{x}+\vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$
- $[c\vec{x}]_{\mathcal{B}} = c[\vec{x}]_{\mathcal{B}}$

This linear transf. is 1-1 and its range is all of $\mathbb{R}^n$.

A linear transf. $V \longrightarrow W$ that's 1-1 and has range all of W is called an <u>isomorphism</u>. If you have one, it means V and W are <u>indistinguishable</u> as vector spaces: <u>any</u> linear alg. calculation you do in V has a mirror rep. in W and vice versa.

Here, the coord. mapping $V \longrightarrow \mathbb{R}^n$ is giving an <u>isomorphism</u> between V and $\mathbb{R}^n$.

<u>Ex</u> Is $\{1+2t^2, 4+t+5t^2, 3+2t\}$ lin. indep. in $\mathbb{P}_2$?

Use the coord. mapping attached to basis $\mathcal{B} = \{1, t, t^2\}$ of $\mathbb{P}_2$: an <u>isomorphism</u> between $\mathbb{P}_2$ and $\mathbb{R}^3$, relates this to the question,

Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \right\}$ lin. indep. in $\mathbb{R}^3$?

[Answer: yes]