Lecture 17 | 26 Oct 2010

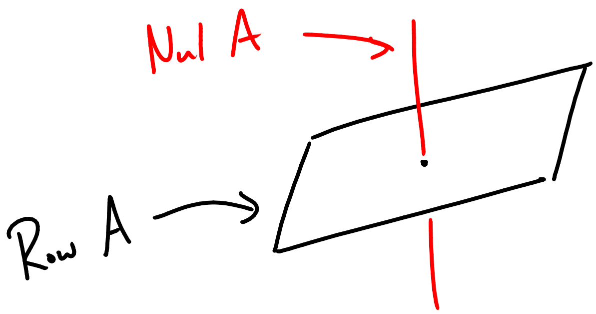
HW6 graded
HW7 back Thu

HW8 due Thu
HW9 due next Tue! (short)
Exam 2 next Thu

Last time: rank of a matrix: if A is $m \times n$ matrix

$$\text{rank } A = \dim \text{Col } A = \dim \text{Row } A = \#\text{ pivots in REF of } A$$

$$\text{rank } A + \dim \text{Nul } A = n$$



$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$$

Ex Say A is $n \times n$ matrix

rank$(A) = n$ if and only if A is invertible

rank$(A) = 0$ if and only if A is the zero matrix

Ex Say A is a 3×7 matrix

rank(A) is one of $0, 1, 2, 3$

dim Nul(A) is one of $7, 6, 5, 4$

Ex

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 3 & 6 & 8 \\ 2 & 7 & -7 & 2 \end{bmatrix}.$$

We can easily see that the rows of A are lin dep.

So $\dim \text{Row } A \leq 3$.

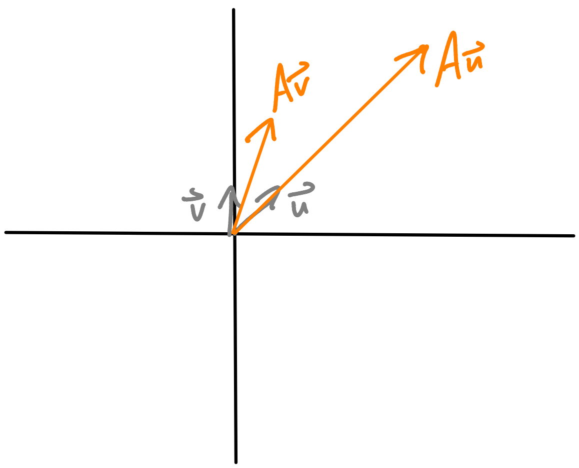
So rank $A \leq 3$.

That implies in particular that the columns of A are also lin dep.

Eigenvectors and Eigenvalues (Sec 5.1)

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A\vec{u} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \qquad A\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



$A\vec{u} = 4\vec{u}$

[So e.g. $A^{1000}\vec{u} = 4^{1000}\vec{u}$
while $A^{1000}\vec{v}$ is really hard to get...]

We say $\vec{u}$ is an eigenvector of A, with eigenvalue 4.

Generally: Say A is an $n \times n$ matrix. Then an eigenvector for A is a vector $\vec{x} \in \mathbb{R}^n$ with $\vec{x} \neq \vec{0}$, $A\vec{x} = \lambda\vec{x}$ for some scalar λ. λ is called the eigenvalue of A corresp to $\vec{x}$.

Ex

$$A = \begin{bmatrix} -2 & -8 & 11 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Is $\vec{v}$ an eigenvector of A? If so, with what eigenvalue?

Just calculate:

$$A\vec{v} = \begin{bmatrix} -3 \\ -1 \\ -1 \end{bmatrix} = (-1)\vec{v}.$$ So $\vec{v}$ is an eigenvector of A, w/ eigenvalue $\lambda = -1$.

[But if instead we take $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ then $A\vec{w} = \begin{bmatrix} 15 \\ 2 \\ 4 \end{bmatrix}$ so $\vec{w}$ is <u>not</u> eigenvector of A]

NB: If $\vec{v}$ is an eigenvector of A w/ eigenval. λ

$$A\vec{v} = \lambda\vec{v}$$

then $c\vec{v}$ is <u>also</u> an eigenvector of A w/ eigenval. λ

$$A(c\vec{v}) = c(A\vec{v}) = c\cdot\lambda\vec{v} = \lambda\cdot(c\vec{v})$$

$$A = \begin{bmatrix} -2 & -8 & 11 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

We just saw that $\lambda = -1$ is an eigenvalue for A.

Is $\lambda = 2$ an eigenvalue for A?

To find out: want to solve $A\vec{x} = 2\vec{x}$

i.e. $A\vec{x} - 2\vec{x} = \vec{0}$

$$(A - 2I)\vec{x} = \vec{0}$$

Does it have nontriv. sol?

$$A - 2I = \begin{bmatrix} -4 & -8 & 11 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

<u>not invertible</u> so $(A-2I)\vec{x} = \vec{0}$ indeed <u>has</u> nontriv. solution

So A <u>does</u> have an eigenvector w/ eigenvalue 2.

Let's find the eigenvector:

$$A-2I \sim \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -5/4 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 - \frac{1}{4}x_3 = 0 \\ x_2 - \frac{5}{4}x_3 = 0 \\ x_3 \text{ free} \end{matrix} \qquad \begin{matrix} 4x_1 - x_3 = 0 \\ 4x_2 - 5x_3 = 0 \end{matrix}$$

ie $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1/4 \\ 5/4 \\ 1 \end{bmatrix} = \frac{1}{4}x_3 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$

So any multiple of $\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$ is an eigenvector of A w/ eigenval. 2

(and vice versa)

Another way of organizing this answer:

Say the λ-eigenspace of A is the subspace of $\mathbb{R}^n$ consisting of all eigenvectors of A w/ eigenvalue λ.

[Have to check it's a subspace: adding 2 eigenvectors gives another eigenvect; multiplying an eigenvector by const. gives another eigenvect]

We just found that in our example the 2-eigenspace of A is $\mathrm{Span}\left\{\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}\right\}$.

Fact: The λ-eigenspace of A is the same as $\mathrm{Nul}(A-\lambda I)$.

$$A\vec{x} = \lambda\vec{x}$$
$$(A-\lambda I)\vec{x} = \vec{0}$$

Ex
$$A = \begin{bmatrix} 5 & 0 & 4 \\ -19/2 & 3 & -19 \\ -3/2 & 0 & 0 \end{bmatrix}$$

What is the 3-eigenspace of A?

It's $\mathrm{Nul}(A-3I)$. $\quad A-3I = \begin{bmatrix} 2 & 0 & 4 \\ -19/2 & 0 & -19 \\ -3/2 & 0 & -3 \end{bmatrix}$

Solve $(A-3I)\vec{v}=\vec{0}$:

$$A-3I \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 = -2x_3 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{matrix}$$

$$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

So $\mathrm{Nul}(A-3I)$ has basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

ie the 3-eigenspace of A has basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$
(it's 2-dimensional)

Ex

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

What is the 5-eigenspace of A?

It's $\mathrm{Nul}(A-5I)$

But $A-5I$ is the zero matrix, O.

The null space of the zero matrix O is the set of all vectors $\vec{x}$ such that $O\vec{x}=\vec{0}$. But that's true for any vector $\vec{x}$!

So $\mathrm{Nul}(A-5I) = \mathbb{R}^4$.

Or:
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 \text{ free} \\ x_2 \text{ free} \\ x_3 \text{ free} \\ x_4 \text{ free} \end{matrix}$$

so $\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Characteristic Equation (Sec 5.2)

How to find the eigenvalues of a matrix A?

λ is an eigenvalue of A if and only if the eq. $(A-\lambda I)\vec{x}=\vec{0}$ has a nontrivial solution. So,

A real # λ is an eigenvalue of A if and only if $\det(A-\lambda I)=0$.

Call the equation $\det(A-\lambda I)=0$ the characteristic equation of A.

Ex $A=\begin{bmatrix}3&1\\1&3\end{bmatrix}$ $\quad A-\lambda I=\begin{bmatrix}3-\lambda&1\\1&3-\lambda\end{bmatrix}$

$$\begin{aligned}\det(A-\lambda I) &= (3-\lambda)(3-\lambda)-1\cdot 1\\ &= 9-6\lambda+\lambda^2-1\\ &= \lambda^2-6\lambda+8\\ &= (\lambda-2)(\lambda-4)\end{aligned}$$

So the eigenvalues of A are 2 and 4.

[We already found the eigenvector $\vec{v}=\begin{bmatrix}1\\1\end{bmatrix}$ w/ eigenvalue 4.
The other eigenvalue comes from $\vec{w}=\begin{bmatrix}1\\-1\end{bmatrix}$ w/ eigenvalue 2.]

Q: What if the roots of char. eq. are complex?
A: Then they don't correspond to eigenvalues as we defined them. (so far.)

Ex

$$A = \begin{bmatrix} 6 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{bmatrix}$$

What are eigenvals. of A?

$$A - \lambda I = \begin{bmatrix} 6-\lambda & 1 & 4 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & -3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (6-\lambda) \begin{vmatrix} 2-\lambda & 3 \\ 0 & -3-\lambda \end{vmatrix}$$

$$= (6-\lambda)(2-\lambda)(-3-\lambda)$$

So the eigenvalues of A are $6, 2, -3$.

More generally –

Fact: The eigenvalues of any upper-triangular matrix are equal to the diagonal entries.