Lecture 20 — 19 Nov 2010

Exam post-mortem:

5) $W = \mathbb{P}_6$

$U : W \to W \qquad U(p) = p''$

a) Find a basis for Ker U. What is the dimension of Ker U?

$$\text{Ker } U = \{\text{all } p \text{ such that } U(p) = 0\}.$$
$$= \{\text{all } p = a + bt \text{ for any } a, b\}$$
$$= \{\text{all } p = a \cdot 1 + b \cdot t \text{ for any } a, b\}$$
$$= \text{Span}\{1, t\}$$

So a basis for Ker U is $\{1, t\}$

$\dim \text{Ker } U = 2$

[Some ppl wrote the basis as $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ — almost right but depends on your choice of basis for $\mathbb{P}_6$

Some ppl wrote Ker $U = \{1, t\}$ — strictly speaking, wrong]

b) Find a basis for Ran U. What is dim Ran U?

$$\dim \text{Ker } U + \dim \text{Ran } U = \dim W$$
$$2 + \dim \text{Ran } U = 7$$
so $\dim \text{Ran } U = 5$

Ran U = {all poly. q in $\mathbb{P}_6$ such that $q = U(p)$ for some p in $\mathbb{P}_6$}

If $p = a + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6$

then $U(p) = 2c + 6dt + 12et^2 + 20ft^3 + 30gt^4$

so Ran $U = \text{Span}\{1, t, t^2, t^3, t^4\}$

and these are lin indep, so

basis for Ran $U = \{1, t, t^2, t^3, t^4\}$

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6)
$$A = \begin{bmatrix} 5 & -3 & 0 \\ 6 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

a) Find the eigenvalues of A.

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -3 & 0 \\ 6 & -4-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda) \begin{vmatrix} 5-\lambda & -3 \\ 6 & -4-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left((5-\lambda)(-4-\lambda) + 18\right)$$

$$= (2-\lambda)(\lambda^2 - \lambda - 2)$$

$$= (2-\lambda)(\lambda+1)(\lambda-2)$$

7h) $\vec{v}$

$$A\vec{v} = \lambda \vec{v}$$
$$B\vec{v} = \mu \vec{v}$$

$$(AB)\vec{v} = A(B\vec{v}) = A(\mu\vec{v})$$
$$= \mu \cdot A\vec{v}$$
$$= \mu \cdot \lambda \vec{v}$$
$$= (\lambda\mu)\vec{v}$$

<u>Fibonacci sequence</u>

$$x_0 = 0$$
$$x_1 = 1$$
$$x_{n+2} = x_n + x_{n+1} \qquad (n \geq 0)$$

$$\{x_n\} = 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

How to find a <u>formula</u> for x_n w/o calculating all the previous terms?

Re-express the recursion as:

$$\vec{v}_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} \qquad \vec{v}_{n+1} = A\vec{v}_n \quad \text{where } A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\left[\begin{pmatrix} x_{n+1} \\ x_{n+2} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} \right]$$

We have $\vec{v}_0 = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{v}_n = A^n \vec{v}_0$$

So finding a formula for x_n $\Longleftrightarrow$ finding a formula for the top entry of $\vec{v}_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$

$\Longleftrightarrow$ finding a formula for the upper right entry of A^n

$$A^n = \begin{bmatrix} * & x_n \\ * & x_{n+1} \end{bmatrix} \qquad A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$$

OK, so <u>how to find A^n?</u>

First, <u>diagonalize</u> A.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= \lambda^2 - \lambda - 1$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

So the 2 eigenvalues are $\frac{1}{2}(1+\sqrt{5})$, $\frac{1}{2}(1-\sqrt{5})$.

<u>Distinct</u> $\Longrightarrow$ we'll get 2 1-dimensional eigenspaces.

$\lambda = \frac{1}{2}(1+\sqrt{5})$: solve $(A-\lambda I)\vec{v}_1 = 0$

get $\vec{v}_1 = c\begin{bmatrix}\sqrt{5}-1\\ 2\end{bmatrix}$

$\lambda = \frac{1}{2}(1-\sqrt{5})$: get $\vec{v}_2 = c\begin{bmatrix}-\sqrt{5}-1\\ 2\end{bmatrix}$

Diagonalize: $P = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix} = \begin{bmatrix}\sqrt{5}-1 & -\sqrt{5}-1\\ 2 & 2\end{bmatrix}$

$$D = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}(1+\sqrt{5}) & 0\\ 0 & \frac{1}{2}(1-\sqrt{5})\end{bmatrix}$$

$A = PDP^{-1}$ $\qquad P^{-1} = \frac{1}{4\sqrt{5}}\begin{bmatrix}2 & 1+\sqrt{5}\\ -2 & \sqrt{5}-1\end{bmatrix}$

$$A^n = PD^nP^{-1} = P\cdot\begin{bmatrix}\left(\frac{1}{2}(1+\sqrt{5})\right)^n & 0\\ 0 & \left(\frac{1}{2}(1-\sqrt{5})\right)^n\end{bmatrix}\cdot P^{-1}$$

$$= \begin{bmatrix} * & \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)\\ * & * \end{bmatrix}$$

So $x_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$!

Could apply this to other recursions like

$$x_{n+3} = 4x_n + 2x_{n+1} - 7x_{n+2}$$

$$\left(\text{involve matrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & -7 \end{bmatrix} \right)$$

That was a (complicated) example of a <u>discrete</u> <u>dynamical</u> <u>system</u>.
Eigenvalues played key role.
Let's look at a few more:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad \vec{x}_n = A\vec{x}_{n-1} = A^n \vec{x}_0$$