Lecture 23

Last time: $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

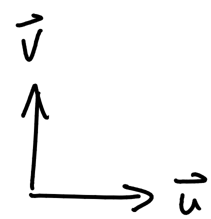
$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$

$$\text{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$$

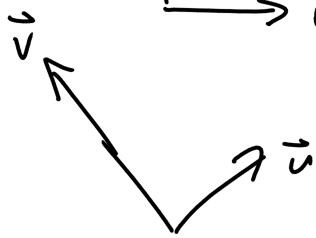
Orthogonality

Call the vectors $\vec{u}, \vec{v}$ orthogonal if $\vec{u}\cdot\vec{v} = 0$.



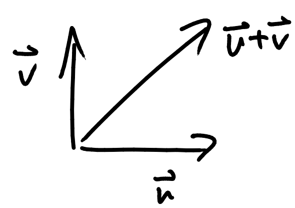
Ex $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are orthogonal.

$\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ " "



Fact $\vec{u}$ and $\vec{v}$ are orthog. if and only if

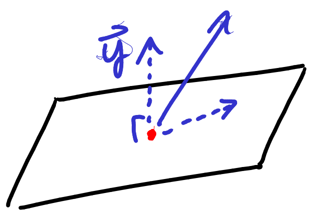
$$\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$$



Why?

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\
&= \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2 \text{ if and only if } \vec{u}\cdot\vec{v} = 0
\end{aligned}$$

Orthogonal complement



Given a plane W in $\mathbb{R}^3$ and a vector $\vec{v}$ in $\mathbb{R}^3$ we sometimes want to decompose $\vec{v}$ into its "part along W" and its "part orthogonal to W"

If W is any subspace of $\mathbb{R}^n$, and $\vec{y} \in \mathbb{R}^n$, we say $\vec{y}$ is <u>orthogonal to W</u> if $\vec{y}$ is orthogonal to <u>every</u> vector $\vec{w}$ in W.

<u>Ex</u> $W = \left\{ \begin{bmatrix} y \\ y \\ z \end{bmatrix} : y, z \text{ any real numbers} \right\}$ subspace of $\mathbb{R}^3$ (2-dimensional)

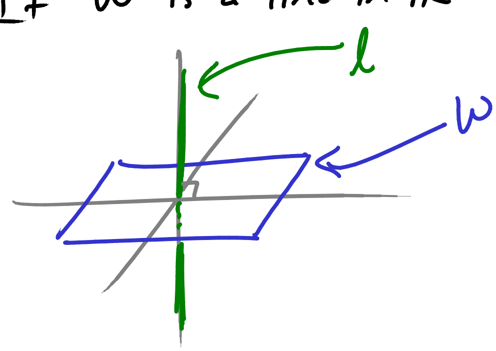
$\vec{y} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is <u>orthogonal to W</u>

Why? Because $\vec{y} \cdot \vec{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ y \\ z \end{bmatrix} = y - y + 0 = \underline{\underline{0}}$

So: given a subspace W of $\mathbb{R}^n$, define the <u>orthogonal complement</u> $W^\perp$ to be the set of all vectors in $\mathbb{R}^n$ which are <u>orthogonal to W</u>.

<u>Ex</u> If W is a plane in $\mathbb{R}^3$ then $W^\perp$ is a line in $\mathbb{R}^3$

If W is a line in $\mathbb{R}^3$ then $W^\perp$ is a plane in $\mathbb{R}^3$



$W^\perp = \ell$

$\ell^\perp = W$

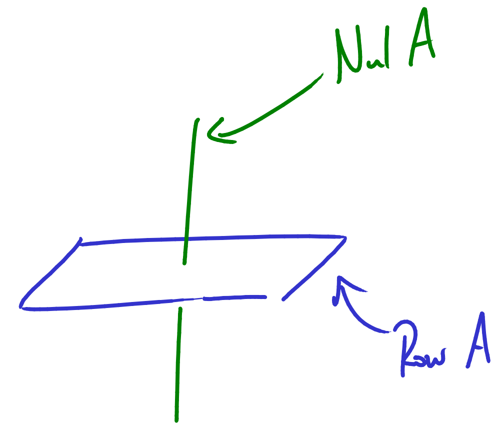
$$W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \right\} \qquad \ell = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \right\}$$

Fact If W is any subspace of $\mathbb{R}^n$,

- $W^\perp$ is a subspace of $\mathbb{R}^n$
- $\dim W + \dim W^\perp = n$

Fact If A is an $m \times n$ matrix,

- $(\text{Row } A)^\perp = \text{Nul } A$
- $(\text{Col } A)^\perp = \text{Nul } A^T$



Why? First let's see why $(\text{Row } A)^\perp = \text{Nul } A$.

Concretely, look at an example:

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad \text{Row } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

To solve $A\vec{x} = \vec{0}$:
$$\left. \begin{array}{l} x_1 - 2x_4 = 0 \\ x_2 + x_4 = 0 \\ x_3 \text{ free} \\ x_4 \text{ free} \end{array} \right] \Rightarrow \text{Nul } A = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

So e.g. $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ is in Nul A: $\begin{bmatrix} 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

To check that e.g. $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ is in Nul A, what did we really do?

Writing $A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \end{bmatrix}$, $A\vec{x} = \begin{bmatrix} \vec{x}\cdot\vec{r}_1 \\ \vec{x}\cdot\vec{r}_2 \end{bmatrix}$

So $A\vec{x} = \vec{0}$ is the same as saying $\vec{x}$ is orthogonal to both $\vec{r}_1$ and $\vec{r}_2$. But that is also the same as saying $\vec{x}$ is orthogonal to any lin. comb. of $\vec{r}_1$ and $\vec{r}_2$:

$$\vec{x}\cdot\left(c_1\vec{r}_1 + c_2\vec{r}_2\right) = c_1\left(\vec{x}\cdot\vec{r}_1\right) + c_2\left(\vec{x}\cdot\vec{r}_2\right)$$
$$= 0 + 0 = 0$$

ie, $\vec{x}$ is orthogonal to Row A.

In sum: $\vec{x} \in \text{Nul } A$ if and only if $\vec{x}$ is orth. to Row A

i.e. $\vec{x} \in (\text{Row } A)^{\perp}$

i.e. $\text{Nul } A = (\text{Row } A)^{\perp}$.

Orthogonal Sets

Call $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^n$ an orthogonal set

if $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$ and all $\vec{v}_i \neq \vec{0}$.

Ex $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix} \right\}$ is an orthogonal set because $\begin{bmatrix}1\\2\\2\end{bmatrix}\cdot\begin{bmatrix}4\\-2\\0\end{bmatrix}=0$, $\begin{bmatrix}1\\2\\2\end{bmatrix}\cdot\begin{bmatrix}2\\4\\-5\end{bmatrix}=0$, $\begin{bmatrix}4\\-2\\0\end{bmatrix}\cdot\begin{bmatrix}2\\4\\-5\end{bmatrix}=0$.

Fact If $\{\vec{v}_1,\dots,\vec{v}_k\}$ is an orthogonal set, then $\{\vec{v}_1,\dots,\vec{v}_k\}$ is linearly indep.

Why? Say $c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_k\vec{v}_k=\vec{0}$ $(*)$

Dot both sides with $\vec{v}_1$: get

$$c_1(\vec{v}_1\cdot\vec{v}_1)+c_2(\vec{v}_2\cdot\vec{v}_1)+\cdots+c_k(\vec{v}_k\cdot\vec{v}_1)=\vec{0}\cdot\vec{v}_1$$

$$c_1(\vec{v}_1\cdot\vec{v}_1)+0+\cdots+0=0$$

i.e. $c_1\|\vec{v}_1\|^2=0$

but $\|\vec{v}_1\|^2\neq 0$, so $c_1=0$

Similarly can see that all the $c_j=0$

So $(*)$ has only the trivial sol^n $\Rightarrow$ $\{\vec{v}_1,\dots,\vec{v}_k\}$ is lin. indep.

So, if $\{\vec{v}_1,\dots,\vec{v}_n\}$ is an orthog. set of n vectors in $\mathbb{R}^n$ then it is a basis. Call it an orthogonal basis.

Orthog. bases are especially convenient:

Fact If $\{\vec{v}_1,\dots,\vec{v}_n\}$ is an orthogonal basis for $\mathbb{R}^n$ and

$$\vec{y}=c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_n\vec{v}_n \qquad (*)$$

then

$$c_j=\frac{\vec{y}\cdot\vec{v}_j}{\|\vec{v}_j\|^2}$$

[Why? Just dot both sides of $(*)$ with $\vec{v}_j$]

<u>Much easier</u> than finding the c_j if $\{\vec{v}_1, \dots, \vec{v}_n\}$ were <u>not</u> orthogonal!

<u>Ex</u> $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is an orthog. basis of $\mathbb{R}^2$.

Express $\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a lin. comb. of $\vec{v}_1$ and $\vec{v}_2$.

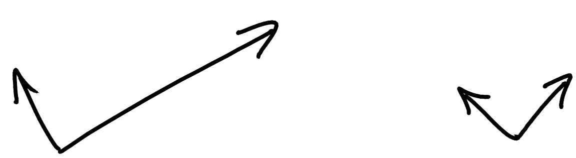
$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2$$

$$c_1 = \frac{\vec{y} \cdot \vec{v}_1}{\|\vec{v}_1\|^2} = \frac{1}{5} \qquad c_2 = \frac{\vec{y} \cdot \vec{v}_2}{\|\vec{v}_2\|^2} = -\frac{4}{20} = -\frac{1}{5}$$

So $\underline{\underline{\vec{y} = \frac{1}{5}\vec{v}_1 - \frac{1}{5}\vec{v}_2}}$ $\left(\text{i.e. } [\vec{y}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{5} \\ -\frac{1}{5} \end{bmatrix}\right)$

Even better:

Say $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an <u>orthonormal basis</u> for $\mathbb{R}^n$ if it is an orthogonal basis <u>and</u> all $\|\vec{v}_i\| = 1$.



orthog. basis orthonormal basis

Ex $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis of $\mathbb{R}^3$.

(Similarly, standard basis is an orthonormal basis of $\mathbb{R}^n$, any n)

Ex $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Ex $\left\{ \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{20}} \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{45}} \begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix} \right\}$

If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal basis and $\vec{y} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

then $c_j = \vec{y} \cdot \vec{v}_j$.

Fact An $m \times n$ matrix A has orthonormal columns if and only if $A^TA = I$.

Why? Ex if A is $m \times 2$: $A = \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}}_{2} \}m \quad A^T = \overbrace{\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{bmatrix}}^{m} \}2$

$$A^TA = \begin{bmatrix} \vec{v}_1^T\vec{v}_1 & \vec{v}_1^T\vec{v}_2 \\ \vec{v}_2^T\vec{v}_1 & \vec{v}_2^T\vec{v}_2 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 \end{bmatrix}$$

$\{\vec{v}_1, \vec{v}_2\}$ orthonormal <u>means</u> $\vec{v}_1 \cdot \vec{v}_1 = 1$ $\quad \vec{v}_1 \cdot \vec{v}_2 = 0$ $\quad \vec{v}_2 \cdot \vec{v}_2 = 1$

$\vec{v}_2 \cdot \vec{v}_1 = 0$

i.e. $A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$