Lecture 6 | 14 Sep 2010

Housekeeping: If you didn't pick up the graded HW1 last class, you can get it at my office hr right after this class, <u>or</u> at the TA's office hrs (any subsequent)

Last time: <u>linear transformations</u>

$$T: \mathbb{R}^n \to \mathbb{R}^m$$
$$T(\vec{x}) + T(\vec{y}) = T(\vec{x}+\vec{y})$$
$$T(c\vec{x}) = cT(\vec{x})$$

We saw that every $T: \mathbb{R}^2 \to \mathbb{R}^m$ is of the form $T(\vec{x}) = A\vec{x}$ for some $m \times 2$ matrix A.

Similarly every $T: \mathbb{R}^n \to \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$ for some $m \times n$ matrix A. (Text calls A "the standard matrix of T")

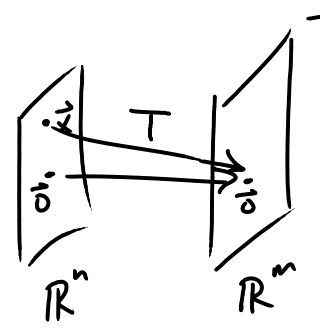
The columns of A are
$$T\Bigg(\underbrace{\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}}_{e_1}\Bigg),\ T\Bigg(\underbrace{\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}}_{e_2}\Bigg),\ T\Bigg(\underbrace{\begin{bmatrix}0\\0\\1\\0\\\vdots\\0\end{bmatrix}}_{e_3}\Bigg),\ \ldots,\ T\Bigg(\underbrace{\begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}}_{e_n}\Bigg).$$

<u>**Fact**</u> $T: \mathbb{R}^n \to \mathbb{R}^m$ is 1-1

$$\Updownarrow$$

$T(\vec{x}) = \vec{0}$ has only the trivial solution

Why? If $T(\vec{x}) = \vec{0}$ had a <u>nontrivial</u> sol^n we see T is not 1-1



And if T is not 1-1 then
$$T(\vec{x}_1) = \vec{b}$$
$$T(\vec{x}_2) = \vec{b}$$
for <u>some</u> $\vec{x}_1 \neq \vec{x}_2$ and some $\vec{b}$.

Then $T(\vec{x}_1 - \vec{x}_2) = T(\vec{x}_1) - T(\vec{x}_2) = \vec{b} - \vec{b} = \vec{0}$

So $\vec{x}_1 - \vec{x}_2$ is a nontriv. solution

Fact: Let A be the standard matrix representing $T: \mathbb{R}^n \to \mathbb{R}^m$

- T has image $\mathbb{R}^m \iff$ columns of A span $\mathbb{R}^m$
- T is 1-1 $\iff$ columns of A are <u>linearly independent</u>

<u>Linear Recurrences</u> (Sec 1.10)

Suppose we have a system whose parameters are measured at discrete time intervals, giving vectors

$$\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots$$

And suppose $\vec{x}_{i+1}$ is obtained from $\vec{x}_i$ by a linear transformation:

$$\vec{x}_{i+1} = A\vec{x}_i$$

This is called a <u>linear recurrence</u>.

<u>Ex</u> $\boxed{\text{City}}$ $\boxed{\text{Suburbs}}$

Say every year 5% of ppl in city move to suburbs
" " 3% " " " sub. " " city

Say r_i = # ppl in city in year $2000+i$
s_i = " " " suburbs " "

Then
$$r_{i+1} = 0.95 r_i + 0.03 s_i$$
$$s_{i+1} = 0.05 r_i + 0.97 s_i$$

Make $\vec{x}_i = \begin{bmatrix} r_i \\ s_i \end{bmatrix}$ Then $\vec{x}_{i+1} = \begin{bmatrix} 0.95r_i + 0.03\,s_i \\ 0.05r_i + 0.97\,s_i \end{bmatrix}$

$$= \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \underbrace{\begin{bmatrix} r_i \\ s_i \end{bmatrix}}_{\vec{x}}$$

i.e. $\vec{x}_{i+1} = A\vec{x}_i$

where $A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$

<u>Ex</u> Fibonacci sequence

$1, 1, 2, 3, 5, 8, 13, \ldots$ $\qquad b_{i+2} = b_{i+1} + b_i$ $\qquad b_1 = 1$, $b_2 = 1$

i.e. $\begin{bmatrix} b_{i+1} \\ b_{i+2} \end{bmatrix} = \begin{bmatrix} b_{i+1} \\ b_{i+1} + b_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_i \\ b_{i+1} \end{bmatrix}$

ie $\vec{x}_{i+1} = A\vec{x}$ where $\vec{x}_i = \begin{bmatrix} b_i \\ b_{i+1} \end{bmatrix}$ $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

So for example — suppose we want to know what happens to a linear recurrence after 4 time steps..

$$\vec{x}_{i+4} = A(A(A(A(\vec{x}_i))))$$

How to represent this <u>more efficiently</u>?

Matrix Operations (Sec 2.1)

Some notation:

$$A = \left.\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ cols}}\right\} m \text{ rows}$$

Zero matrix O has all entries 0 ($a_{ij} = 0$ for all i,j)

A square matrix is one with $m = n$. Then its diagonal entries are the a_{ii}, i.e. $a_{11}, a_{22}, \ldots, a_{nn}$

$$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & a_{33} & \\ & & & \ddots \; a_{nn} \end{bmatrix}$$

A diagonal matrix is one whose non-diagonal entries are all zero

Ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a diagonal matrix. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is also diag. (identity matrix)

Addition of matrices: If A, B are 2 matrices of the same size ($m \times n$) then define $A+B$ to be another $m \times n$ matrix: its entries are just $(A+B)_{ij} = A_{ij} + B_{ij}$

If A, B are of different sizes then $A+B$ is not defined

Ex $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ 4 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -7 & 6 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 4 & 8 \\ 0 & 2 \end{bmatrix}$

$$A+B=\begin{bmatrix}-4 & 8\\ 2 & 4\\ 5 & 3\end{bmatrix} \qquad A+C \text{ is not defined}$$

<u>Scalar multiplication</u> If A is a matrix $(m\times n)$ and c is a constant

cA is another $m\times n$ matrix

$$(cA)_{ij} = c\cdot A_{ij}$$

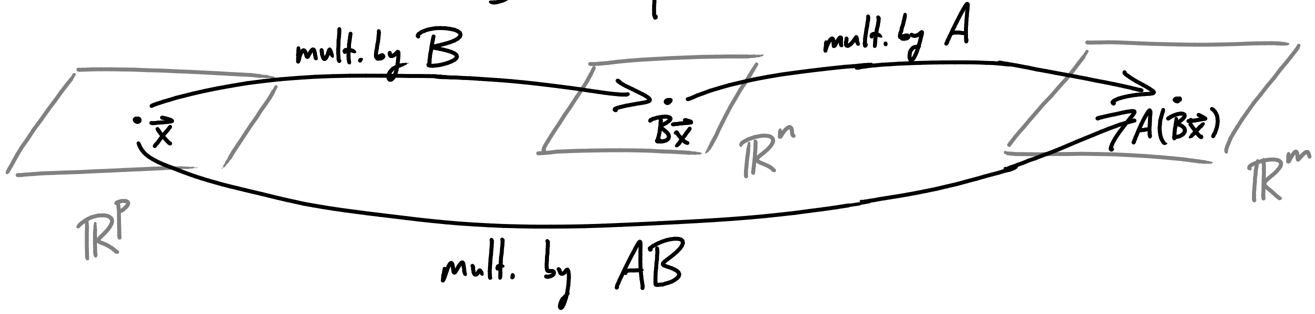
<u>Ex</u> $A=\begin{bmatrix}3 & 1\\ 1 & 4\end{bmatrix} \qquad 4A=\begin{bmatrix}12 & 4\\ 4 & 16\end{bmatrix}$

$$-A=(-1)A=\begin{bmatrix}-3 & -1\\ -1 & -4\end{bmatrix}$$

<u>Fact</u>:

- $A+B=B+A$
- $(A+B)+C=A+(B+C)$
- $A+0=A$
- $r(A+B)=rA+rB$
- $(r+s)A=rA+sA$
- $r(sA)=rsA$

<u>Matrix Multiplication</u> Say: A is $m\times n$
B is $n\times p$



We want a matrix AB such that $(AB)(\vec{x}) = A(B(\vec{x}))$

What is it?

Take $B=\begin{bmatrix}\vec{b}_1 \cdots \vec{b}_p\end{bmatrix} \qquad \vec{x}=\begin{bmatrix}x_1\\ \vdots\\ x_p\end{bmatrix}$

$$B\vec{x} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_p\vec{b}_p$$

$$\begin{aligned} A(B\vec{x}) &= A(x_1\vec{b}_1 + \cdots + x_p\vec{b}_p) \\ &= x_1 A(\vec{b}_1) + x_2 A(\vec{b}_2) + \cdots + x_p A(\vec{b}_p) \\ &= x_1 (A\vec{b}_1) + x_2 (A\vec{b}_2) + \cdots + x_p (A\vec{b}_p) \\ &= \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \end{aligned}$$

So, we should define $AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \end{bmatrix}$

("AB is the matrix obtained by multiplying each column of B by A")

AB is an $m \times p$ matrix.

$$\underset{(m\times n)}{A} \times \underset{(n\times p)}{B} = \underset{(m\times p)}{AB}$$

Ex $A = \begin{bmatrix} 1 & -2 \\ 4 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{array}$$

$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 7 & 15 & -19 \end{bmatrix}$$

Note: the def. of AB only makes sense if # columns of A = # rows of B.

Ex If A is 6×4 and B is 4×5 then AB is defined, BA is not defined

Warning: Even if AB and BA are both defined (A,B both $n\times n$) AB and BA might not be equal!

Ex $A=\begin{bmatrix}1&0\\0&4\end{bmatrix}$ $B=\begin{bmatrix}1&1\\1&1\end{bmatrix}$

$AB=\begin{bmatrix}1&1\\4&4\end{bmatrix}$ $BA=\begin{bmatrix}4&1\\4&1\end{bmatrix}$ so $AB\neq BA$!

"Row-column rule" for matrix multiplication:

$$(AB)_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+a_{i3}b_{3j}+\cdots+a_{in}b_{nj}$$

Ex $A=\begin{bmatrix}1&-2&4\\2&0&3\end{bmatrix}$ (go across) $B=\begin{bmatrix}3&-1\\0&-1\\5&-1\end{bmatrix}$ (go down)

$$AB=\begin{bmatrix}1\cdot3+(-2)\cdot0+4\cdot5 & 1\cdot(-1)+(-2)\cdot(-1)+4\cdot(-1)\\2\cdot3+0\cdot0+3\cdot5 & 2\cdot(-1)+0\cdot(-1)+3\cdot(-1)\end{bmatrix}$$

$$=\begin{bmatrix}23&-3\\21&-5\end{bmatrix}$$

(Sizes: $\overset{A}{(2\times3)}\times\overset{B}{(3\times2)}=\overset{AB}{2\times2}$)

Facts

- $A(BC) = (AB)C$
- $A(B+C) = AB + AC$
- $(B+C)A = BA + CA$
- $r\cdot(AB) = (rA)B = A(rB)$
- $I \cdot A = A = A \cdot I$ (I = identity matrix)

Warning: If $AB = AC$
and $A \neq 0$
it might __not__ be true that $B = C$.

Warning: If $AB = 0$
it might __not__ be true that $A = 0$ or $B = 0$.

Ex $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

[A = projection on x-axis
B = projection on y-axis
so AB = projection on y-axis __followed by__ projection on x-axis
and indeed, applying that sequence of operations to any vector $\vec{x}$ gives the zero vector: $(AB)(\vec{x}) = \vec{0}$
i.e. AB is represented by the zero matrix, $O\vec{x} = \vec{0}$]