Lecture 8

21 Sep 2010

Exam 1 (midterm) Sep 30 (next Thu) in class

75 min.

Review next Tue.

1 sheet handwritten notes allowed (front+back)

No calculators

Practice exam will be posted by Thu. along w/ some from previous yrs.

Difficulty comparable to HW problems

Last time: the <u>inverse</u> of a square matrix

A square matrix $\rightsquigarrow A^{-1}$ $\qquad AA^{-1} = I$

(invertible) $\qquad A^{-1}A = I$

Say A is square.

<u>Fact</u>: <u>If</u> there is a matrix C with $AC = I$ <u>then</u> A is invertible and $C = A^{-1}$.

Similarly: <u>if</u> there is a matrix C with $CA = I$ <u>then</u> A is invertible and $C = A^{-1}$.

(in the text Sec 2.2 — key statement is "Theorem 7" — proof involves writing A as a product of "elementary matrices")

Fact If A, B invertible $n \times n$ matrices
Then AB is also invertible

$$(AB)^{-1} = B^{-1}A^{-1}$$

Why? $(AB)(B^{-1}A^{-1}) = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I$

could also check $(B^{-1}A^{-1})(AB) = I$

Similarly $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Fact If A is invertible then A^T is invertible

$$(A^T)^{-1} = (A^{-1})^T$$

(and, If A not invertible then A^T not invertible)

From last time: Algorithm for finding A^{-1}

$$\left[A \mid I \right] \sim \left[I \mid A^{-1} \right]$$

Why does it work?

The formula $AA^{-1} = I$ means that the k^{th} column of A^{-1} is the solution of $A\vec{x} = \vec{e}_k$ where $\vec{e}_k$ is the k^{th} column of I

And to solve that equation, you would do row-reduction of the matrix

$$[A \mid \vec{e}_k].$$

So finding the whole A^{-1} means solving all these eq. at once:

row-reduce

$$[A \mid I].$$

Ex If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$ find the 2^{nd} column of A^{-1}!

That's the same as finding the solution of $A\vec{x} = \vec{e}_2$

where $\vec{e}_2$ is the 2^{nd} col. of I

i.e. $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

So: solve $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4/3 \\ 0 & 1 & 0 & 11/3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$\uparrow$ 2^{nd} column of A^{-1}!

Ex Is $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ invertible?

Yes:
- row reduction $\rightsquigarrow$ 2 pivots
- $ad - bc = 1 \cdot 5 - 3 \cdot 2 = -1 \neq 0$

- the columns are linearly independent

Ex Is $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ invertible?

Yes: - row reduction $\rightsquigarrow$ 3 pivots
($\Rightarrow$ the columns are linearly independent)

What is the inverse?

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$A \cdot A^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex Is $\begin{bmatrix} 2 & 4 & 8 \\ -1 & -2 & -4 \\ 3 & 5 & 11 \end{bmatrix}$ invertible?

No: 1^{st} and 2^{nd} rows are multiples of one another
$\Rightarrow$ rows are linearly dependent
$\Rightarrow$ columns of A^T are lin dep
$\Rightarrow$ A^T not invertible
$\Rightarrow$ A not invertible!

Ex Is the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by rotation by 30° around the x-axis invertible?

Yes: - its inverse is just the opposite rotation (by −30°)

- it's 1-1
- its range is all of $\mathbb{R}^3$

[Standard matrix for this rotation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30° & -\sin 30° \\ 0 & \sin 30° & \cos 30° \end{bmatrix}$]

Ex Is the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x \\ y \end{bmatrix}$$ invertible?

No: - its range is a plane

- the equation $T(\vec{u}) = \vec{0}$ has nontrivial solutions

  such as $\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Ex $A=\begin{bmatrix}1&3\\4&7\end{bmatrix}$ $B=\begin{bmatrix}-2&2\\3&1\end{bmatrix}$ $C=\begin{bmatrix}4&2\\0&0\end{bmatrix}$ $D=\begin{bmatrix}1&6\\3&5\end{bmatrix}$

Is ABD invertible? Yes, it's product of invertible matrices

$$(ABD)^{-1}=D^{-1}B^{-1}A^{-1}$$

Is $ABCD$ invertible? No, it's product including C which is not invertible

So the composition $ABCD(\vec{x})$ can't be 1-1:

Since C is not 1-1, there are 2 diff. vectors $\vec{x},\vec{y}$ with $C(D(\vec{x}))=C(D(\vec{y}))$

Then also $ABCD(\vec{x})=ABCD(\vec{y})$

So $ABCD$ is not 1-1

i.e. $ABCD$ not invertible

Ex

Is $\begin{bmatrix}1&0&0&0\\0&2&-2&0\\0&0&4&0\\0&0&0&3\end{bmatrix}$ invertible?

Yes: has a pivot in every column

Partitioned Matrices (Sec 2.4)

Sometimes it's convenient to think of a matrix as being divided into blocks.

Ex

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0 & -4 \\ 3 & -1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & -4 & -2 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 7 & -1 \end{bmatrix}$$

$$B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix} \qquad B_2 = \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$$

$$\left[\begin{array}{c|c} US \to US & US \to F \\ \hline F \to US & F \to F \end{array}\right]$$

The product can be calculated using the block structure:

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$

$$A_{11}B_1 + A_{12}B_2 = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix} + \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -6 & 2 \end{bmatrix}$$

$$A_{21}B_1 + A_{22}B_2 = \quad -- \quad + \quad -- \quad = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

so $$AB = \left[\begin{array}{cc} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{array}\right]$$

Ex

$$\left[\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & A_2 & 0 \\ \hline 0 & 0 & A_3 \end{array}\right] \left[\begin{array}{c|c|c} B_1 & 0 & 0 \\ \hline 0 & B_2 & 0 \\ \hline 0 & 0 & B_3 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} A_1B_1 & 0 & 0 \\ \hline 0 & A_2B_2 & 0 \\ \hline 0 & 0 & A_3B_3 \end{array}\right]$$